GLOBAL EXISTENCE VIA A MULTIVALUED OPERATOR FOR AN ALLEN-CAHN-GURTIN EQUATION

MICHEL PIERRE AND MORGAN PIERRE

Nous dédions cet article à Jerry Goldstein, avec notre admiration et notre amitié.

ABSTRACT. The main goal of this paper is to prove existence of global solutions in time for an Allen-Cahn-Gurtin model of pseudo-parabolic type. Local solutions were known to "blow up" in some sense in finite time. It is proved that the equation is actually governed by a monotone-like operator. It turns out to be multivalued and measure-valued. The measures are singular with respect to the Lebesgue measure. This operator allows to extend the local solutions globally in time and to fully solve the evolution problem. The asymptotic behavior is also analyzed.

1. INTRODUCTION.

The goal of this paper is to study the question of global existence in time of solutions to evolution equations of the following type

$$u_t + V \cdot \nabla u_t - \Delta u_t + f(u) - \theta u - \alpha \Delta u = 0 \quad x \in \Omega,\ t > 0,\ u(0, \cdot) = u_0, \tag{1}$$

where $V \in \mathbb{R}^N$, θ and α are nonnegative real numbers and $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function. A typical and relevant example in applications is given by

$$(a, b) = (-1, 1),\ f(r) = \ln \frac{1 + r}{1 - r}. \tag{2}$$

We choose to work with *periodic conditions* on the boundary of the open hypercube $\Omega := (0, 1)^N \subset \mathbb{R}^N$.

The model (1) is a version of the sometimes called Allen-Cahn-Gurtin models. The standard Allen-Cahn equation characterizes qualitative features of two-phases systems; it reads

$$u_t - \alpha \Delta u + g(u) = 0 \quad x \in \Omega,\ t > 0, \tag{3}$$

where $g : (a, b) \to \mathbb{R}$ is the derivative of a double-well potential (we typically have $g(r) = f(r) - \theta r$ with f as above and $\theta > 0$ large enough). In this context, u is an order parameter which describes the ordering of atoms within a unit cell on a lattice, and the wells of the potential define the phases of the system. In [10], by considering a local microforce balance, M. E. Gurtin derived several generalizations of the Allen-Cahn equation for anisotropic materials, including equation (1) and its generalization (38) below.

A first look at the equation (1) suggests a natural way to prove existence of a solution. Indeed, if we denote $L = I + V \cdot \nabla - \Delta$, then L defines an isomorphism from $H := H^1_{per}(\Omega)$ *(see below for a precise definition)* into its dual H' so that (1), together with periodic boundary conditions, may be rewritten in the equivalent form

$$u_t + L^{-1} f(u) = \alpha L^{-1} \Delta u + \theta L^{-1} u,\ u(t) \in H,\ t > 0,\ \ u(0, \cdot) = u_0 \in H. \tag{4}$$

Here, $\alpha L^{-1}\Delta + \theta L^{-1}$ is a continuous linear operator from H into itself so that the question of existence depends essentially on the operator $L^{-1}f$ (in other words, the case $\alpha = 0$, $\theta = 0$ is significant of the general problem).

Local existence in time may be obtained in general, at least when f is regular, for regular enough initial data with values in a compact subset of (a, b) (use the standard Cauchy-Lipschitz theorem in an appropriate Sobolev space). But, global existence is a quite more serious question: indeed, the following result has been proved in [6] in the case of the nonlinearity (2) in dimension $N = 1, \Omega = (0, 1)$: for α small enough, there exist $T^* > 0$ and u a unique regular solution of (4) on $[0, T^*]$ such that $\|u(t)\|_{L^\infty(\Omega)} < 1$ for all $t \in [0, T^*)$ and $u(T^*, 1/2) = 1, u_t(T^*, 1/2) > 0$. In particular, this solution u cannot be extended as a standard "regular" solution of (4) for $t > T^*$. The result stated in [6] actually assumes $\theta > 2$, but a similar proof can be applied for any $\theta \in \mathbb{R}$.

A main point here is that no maximum or comparison principle holds for (1), (even for $V = 0$). This may be surprising at first since it does hold for the linear problem ($f(u) = 0, V = 0, \theta = 0$), as proved in [7] in the context of pseudo-parabolic problems. However, the introduction of the nonlinearity f, even if nondecreasing, destroys this maximum principle property. This is different from more classical Allen-Cahn models. Here, if one starts with initial data with values in a compact subset of $(-1, 1)$, then the solution may reach the values 1 or -1 in finite time (and recall we have $|f(1^-)| = |f(-1^+)| = \infty$).

Our main goal here is to describe what happens after time T^* in this kind of situation. Let us first suppose $V = 0$ (with moreover $\theta = 0 = \alpha$). Then, as we prove in Section 2, the operator $[u \to Au := L^{-1}f(u)]$ is monotone in H (for its natural Hilbert scalar product) and the range of $I + \lambda A$ is dense in H for all $\lambda > 0$. Therefore, the closure $\overline{A}$ of A in $H \times H$ is maximal monotone. As a consequence of the theory of maximal monotone operators (see [2]), the problem

$$(5) \qquad u_t + \overline{A}u \ni 0,\ u \in C([0, \infty); H),\ u(0, \cdot) = u_0,$$

has a unique global solution on $[0, \infty)$. This solution necessarily coincides with the local solution of (4) on some interval $[0, T]$: therefore, the global solution of (5) provides an extension of the local solutions of (4) even in "blow up" situations like the one described in [6] and recalled above.

A surprising fact is that, in general, the operator $\overline{A}$ is multivalued (whence the sign "$\ni$" rather than "$=$" in (5)). We prove in Section 2 that, if $w \in \overline{A}u$, then there exists a measure ν on Ω such that $w = L^{-1}(f(u) + \nu)$. This measure writes $\nu = \nu_b - \nu_a$ where ν_a, ν_b are nonnegative measures supported respectively by the sets $[u = a]$ and $[u = b]$. These sets are of Lebesgue measure zero so that ν_a, ν_b are singular with respect to the Lebesgue measure. Then, we may interpret the problem (5) as solving

$$(6) \qquad u_t(t) - \Delta u_t(t) + f(u(t)) + \nu(t) = 0,$$

where the measure ν on $[0, \infty) \times \Omega$ is singular with respect to the Lebesgue measure and carried by the set $[u = a] \cup [u = b]$. This measure is equal to zero in some situations, in which case u solves exactly (1) and (4). This is in particular the case when $(a, b) = (-\infty, +\infty)$. This provides an extension of some results in [5] where global existence was proved in dimension $N \leq 3$ and for polynomial potentials f with subcritical growth.

We chose to devote Section 2 to the particular problem (6), that is to assume first $V = 0, \alpha = 0 = \theta$ in (1). This allows to purely understand the behavior of the measures ν. Explicit examples are given in dimension $N = 1$ which are confirmed by numerical computations given in Subsection 2.4 where this singular behavior may be captured. In this section, $\overline{A}$ turns out to be a subdifferential so that the solutions are rather regular, even if the initial data are taken in the closure of the domain of $\overline{A}$ and may then take the singular values a or b on sets of positive measure.

As explained above, taking $\alpha \neq 0$ does not make much difference since $L^{-1}\Delta$ is a good continuous linear perturbation of $\overline{A}$. But, making $V \neq 0$ changes the analysis quite a bit. First, the problem is not symmetric any more, and can certainly not be governed by a subdifferential operator as for $V = 0$. Then, we even loose the monotonicity property. However, we are able to prove that the evolution problem is still governed by a multivalued operator, which is not monotone, but which satisfies monotonicity-like properties. We exploit them to prove existence of global solutions to the full problem (1). This general problem is treated in Section 3 with even a more general diffusion (see (38)).

Finally, we also analyze the asymptotic behavior, first in the particular case $V = 0, \alpha = 0 = \theta$ (see Subsection 2.3), then in the general case (see Subsection 3.3). It turns out that, in dimensions $N \leq 3$, the solution takes its values in a compact set of (a, b) for large t: therefore equation (1) (or even (38)) is then exactly satisfied no matter if (a, b) is unbounded or not.

2. The particular case $V = 0, \alpha = 0 = \theta$.

2.1. **The governing operator $\overline{A}$.** Let $\Omega = (0,1)^N$ and let $H := H_{per}(\Omega)$ be defined by

$$(7)\qquad \begin{aligned} &H_{per}(\Omega) = \big\{u \in H^1_{loc}(\mathbb{R}^N);\ \forall x = (x_1, ..., x_N) \in \mathbb{R}^N,\ \forall i = 1, ..., N,\\ &u(x_1, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_N) = u(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_N)\big\},\end{aligned}$$

equipped with the Hilbert norm $\|\cdot\|_H$ associated to the scalar product

$$(8)\qquad \forall u, w \in H,\ (u, w)_H = \int_\Omega u\, w + \nabla u \nabla w =< (I - \Delta)u, w >_{H' \times H},$$

where the dual space H' is the subspace of Ω-periodic distributions of $H^{-1}_{loc}(\mathbb{R}^N)$ and $I - \Delta$ defines an isomorphism from H into H'. We denote $J = (I - \Delta)^{-1}$. We have the embedding $H \hookrightarrow L^2(\Omega) \hookrightarrow H'$ and for $v \in L^2(\Omega), w \in H$

$$(9)\qquad \int_\Omega v\, w =< v, w >_{H' \times H} = (Jv, w)_H.$$

Let $-\infty \leq a < b \leq +\infty$ and let $f : (a, b) \to \mathbb{R}$ be maximal monotone continuous that is

$$(10)\qquad \begin{cases} f \text{ is continuous nondecreasing,}\\ [-\infty < a] \Rightarrow [f(a^+) := \lim_{r\to a, r>a} f(r) = -\infty]\,,\\ [b < +\infty] \Rightarrow [f(b^-) := \lim_{r\to b, r<b} f(r) = +\infty]\,. \end{cases}$$

We consider the operator A defined on H by

$$(11)\qquad \begin{cases} D(A) = \{u \in H;\ f(u) \in L^2(\Omega)\},\\ \forall u \in D(A),\ A\, u = J(f(u)). \end{cases}$$

It will be stated in the next theorem that the closure $\overline{A}$ of the operator A in $H \times H$ is a maximal monotone operator, and is even the subdifferential of a convex lower semi-continuous ($l.s.c.$) function $\Phi : H \to (-\infty, \infty]$ (see [2] for more details). To define Φ, we fix some $c \in (a, b)$ and we introduce:

$$(12)\qquad \forall \xi \in (a, b), F(\xi) = \int_c^\xi f(s)ds,\ \forall \xi \in \mathbb{R} \setminus [a, b],\ F(\xi) = +\infty.$$

If $b < +\infty$, then $f(b^-) = +\infty$ so that F is increasing near b and we may define $F(b) = \lim_{\xi\to b, \xi<b} F(\xi) \leq +\infty$. Similarly, if $a > -\infty$, we define $F(a) =$

$\lim_{\xi\to a,\xi>a} F(\xi) \le +\infty$. We easily check that F is convex, *l.s.c.* on $\mathbb{R}$. Next we define for all $u \in H$

$$\Phi(u) := \begin{cases} \int_\Omega F(u)dx & \text{if } F(u) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{13}$$

The domain of Φ is defined as $D(\Phi) := \{u \in H \ : F(u) \in L^1(\Omega)\}$. We easily check that Φ is convex. It is *l.s.c.* on H: indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence converging to u in H and assume that $\lim_{n\to\infty} \Phi(u_n) < +\infty$. Thus, u_n is bounded in H and therefore relatively compact in $L^2(\Omega)$. Up to a subsequence, we may assume that u_n converges almost everywhere to u. In particular, $a.e.x, a \le u(x) \le b$. By continuity of F on (a, b), $F(u_n(x))$ converges *a.e.* to $F(u(x))$ on the set $[a < u < b]$. On the set $[-\infty < a = u]$, either F extends by continuity at a and $F(u_n(x)) \to F(u(x))$ *a.e.*, or $F(a) = +\infty$ and $F(u_n(x)) \to +\infty$ *a.e.*. The analysis is similar near b. Therefore, we may apply Fatou's lemma to conclude that $\int_\Omega F(u) \le \liminf_{n\to\infty} \int_\Omega F(u_n)$ (recall that, by convexity, F is bounded from below by a linear function).

We denote by $\mathcal{M}$ the set of Radon measures on $\mathbb{R}^N$ which are finite on $\overline{\Omega}$ and

$$\forall \mu \in \mathcal{M}, \|\mu\|_{\mathcal{M}} := \sup\{\left|\int_\Omega \varphi d\mu\right|, \ \varphi \in C(\overline{\Omega}), \|\varphi\|_{L^\infty(\Omega)} \le 1\}. \tag{14}$$

Theorem 1. *The closure $\overline{A}$ of A in $H \times H$ satisfies the following*

- *$\overline{A}$ is maximal monotone and $\partial\Phi = \overline{A}$.*
- *there exists $C > 0$ such that*

$$\forall u \in D(\overline{A}), \forall w \in \overline{A}u, \|f(u)\|_{L^1(\Omega)} \le C[\|u + w\|_H^2 + 1], \tag{15}$$

- *for all $u \in D(\overline{A})$, $w \in \overline{A}u$ if and only if there exist two positive finite measures $\nu_a, \nu_b \in \mathcal{M}$ such that*

$$\begin{cases} f(u) \in L^1(\Omega), f(u) + \nu_b - \nu_a \in H', \nu_a([a < u]) = 0 = \nu_b([u < b]), \\ w = J(f(u) + \nu_b - \nu_a). \end{cases} \tag{16}$$

Remark 1. If $N = 1$, then any function of H is continuous or, more precisely, has a (unique) continuous representation. This is no longer true in dimension $N \ge 2$ and it is replaced by the fact that any function of H has a *quasi-continuous representation* which is unique up to a set of H^1-zero capacity (see the Appendix). Therefore, the set $[a < u < b]$ is defined up to a set of zero capacity. Moreover, a Radon measure which is also in H' does not charge the sets of zero capacity. This property is shared by ν_a, ν_b so that $\nu_a([a < u]), \nu([u < b])$ are well-defined and (16) makes sense. For the reader's convenience, all the necessary definitions and results are recalled in the Appendix.

Note that ν is carried by the set $[u = a] \cup [u = b]$ which is of Lebesgue-measure zero since $|f(u)| = \infty$ on this set and $f(u) \in L^1(\Omega)$. Therefore, ν is singular with respect to the Lebesgue measure. It may be a Dirac mass in dimension $N = 1$ as shown by the examples below (see Remark 3). It is identically zero if $(a, b) = \mathbb{R}$ (see Remark 2 below).

Proof of Theorem 1. The proof is divided into five steps:
Step 1: $A \subset \partial\Phi$; **Step 2:** $\overline{A}$ is maximal monotone (so that $\overline{A} = \partial\Phi$); **Step 3:** the estimate (15); **Step 4:** any $w \in \overline{A}u$ satisfies (16); **Step 5:** for any (u, w) satisfying (16), one has $w \in \overline{A}u$.

Step 1: Let us first check that $A \subset \partial\Phi$. Let $u \in D(A)$: then, $\Phi(u) < +\infty$. Indeed, we may write by convexity of F that

$$(u - c)f(c) \le F(u) - F(c) = F(u) \le (u - c)f(u).$$

Since $u, f(u) \in L^2(\Omega)$, this implies that $F(u) \in L^1(\Omega)$ and $\Phi(u) < +\infty$.

Now, let $v \in H$ with $\Phi(v) < +\infty$. Then

$$\Phi(v) - \Phi(u) = \int_\Omega [F(v) - F(u)] \geq \int_\Omega [v-u] f(u) =< f(u), v-u >_{H' \times H} .$$

This may also be written (see (9))

$$\Phi(v) - \Phi(u) \geq (Jf(u), v-u)_H = (Au, v-u)_H,$$

which implies $u \in D(\partial\Phi)$ and $Au \in \partial\Phi(u)$.

Since $\partial\Phi$ is maximal monotone, and therefore closed, we also have $\overline{A} \subset \partial\Phi$. In particular $\overline{A}$ is monotone.

Step 2: Let us show that $\overline{A}$ is itself maximal monotone: it will follow $\overline{A} = \partial\Phi$. Given $g \in H$, if moreover $g - \Delta g \in L^2(\Omega)$, then, since f is maximal monotone, the problem

$$u \in H, f(u) \in L^2(\Omega), u - \Delta u + f(u) = g - \Delta g \tag{17}$$

has a unique solution (this is classical, but a proof is given in the Appendix for the reader's convenience). The identity (17) is equivalent to the two formulations

$$u \in H, f(u) \in L^2(\Omega),\ u + Jf(u) = g \text{ or } u \in D(A),\ u + Au = g.$$

Now, let $g \in H$; let $(g_n)_{n\geq 0}$ converge to g in H with $g_n - \Delta g_n \in L^2(\Omega)$. Let $u_n = (I+A)^{-1}(g_n)$. By monotonicity of A in H, $(I+A)^{-1} : H \to H$ is non expansive; since (g_n) is a Cauchy sequence, then (u_n) is also a Cauchy sequence. Let u be its limit in H. Then, $Au_n = g_n - u_n$ converges also in H and its limit is $g - u$. By definition of $\overline{A}$, $g - u \in \overline{A}$ and this proves that $I + \overline{A}$ is onto, whence the maximal monotonicity of $\overline{A}$ and $\overline{A} = \partial\Phi$.

Step 3: Let $u \in D(A), w \in \overline{A}u, g = u + w$. Let $(g_n)_{n\geq 0}$ approximating g in H with $g_n - \Delta g_n \in L^2(\Omega)$ and let $u_n = (I+A)^{-1} g_n$ as above. Then, u_n converges to u in H. Recall we chose $c \in (a,b)$ with $F(c) = 0$. Then, multiplying

$$[u_n - \Delta u_n + f(u_n)] - [c + f(c)] = g_n - \Delta g_n - c - f(c)$$

par $u_n - c$, leads to

$$\|u_n - c\|_H^2 + \int_\Omega (f(u_n) - f(c))(u_n - c) \leq \|u_n - c\|_H [\|g_n\|_H + C],$$

where $C = C(c)$ depends only on c. Using Young's inequality: $ab \leq (a^2+b^2)/2$ for the right-hand side leads to

$$\|u_n - c\|_H^2 + 2\int_\Omega (f(u_n) - f(c))(u_n - c) \leq [\|g_n\|_H + C]^2. \tag{18}$$

According to the continuity of f at c, let $\eta > 0$ such that

$$|r - c| \leq \eta \ \Rightarrow\ |f(r) - f(c)| \leq 1.$$

Then, we successively deduce

$$\int_\Omega |f(u_n)| \leq meas(\Omega)|f(c)| + \int_{[|u_n - c| \leq \eta]} [f(u_n) - f(c)| + \int_{[|u_n - c| > \eta]} |f(u_n) - f(c)|.$$

$$\int_\Omega |f(u_n)| \leq meas(\Omega)[|f(c)| + 1] + \eta^{-1} \int_\Omega (u_n - c)\left(f(u_n) - f(c)\right).$$

$$\int_\Omega |f(u_n)| \leq C[1 + \|g_n\|_H^2],$$

where $C = C(c, \Omega, \eta)$. Since u_n converges to u in H, up to a subsequence the convergence holds also a.e. so that, by Fatou's Lemma

$$\int_\Omega |f(u)| \le C[1 + \|g\|_H^2].$$

Step 4: Let us show that w satisfies (16). We use the same notations as in Step 3. Since $f(u_n)$ is bounded in $L^1(\Omega)$, it converges (up to a subsequence) to a bounded measure that we denote $f(u) + \nu$ where $f(u) \in L^1(\Omega)$. This convergence holds in the sense of measures, but also in H' since $f(u_n) = (I - \Delta)(g_n - u_n)$ and, as both convergences imply the convergence in the sense of distributions, the limit is the same. We may also write $u + J[f(u) + \nu] = g$.

In what follows, for any function of H, we always choose its quasi-continuous representation- see the Appendix for all necessary results. Since u_n converges to u in H, up to a subsequence we may assume that $u_n(x)$ converges to $u(x)$ for all x outside a set of capacity zero (we say "quasi-everywhere"). Let $T \in C^1(\mathbb{R}), T \ge 0$, with compact support. Then, $T(u_n)$ converges to $T(u)$ in H and quasi-everywhere. For a test-function $\psi \in C_0^\infty(\Omega)$, and for $\mu = f(u) + \nu$, we have (see (9))

$$\int_\Omega \psi T(u_n) f(u_n) = < f(u_n), \psi T(u_n) >_{H' \times H}$$

that we may also write

$$\int_\Omega \psi T(u_n) f(u_n) = < f(u_n) - \mu, \psi T(u_n) >_{H' \times H} + < \mu, \psi T(u_n) >_{H' \times H} .$$

The first term on the right-hand side tends to zero as $n \to \infty$ since it is bounded above by $\|[f(u_n) - \mu]\|_{H'} \|\psi T(u_n)\|_H$ which tends to zero. Thanks to Lemma 7 in the Appendix, the second term may also be written $\int_\Omega \psi T(u_n) d\mu$. Since $T(u_n)$ converges quasi-everywhere to $T(u)$ and is uniformly bounded, and since μ is a finite measure which does not charge the sets of zero-capacity, by the dominated convergence theorem, this integral tends to $\int_\Omega \psi T(u) d\mu$. Thus, we may claim for all $\psi \in C_0^\infty(\Omega)$

$$\lim_{n\to\infty} \int_\Omega \psi T(u_n) f(u_n) = \int_\Omega \psi T(u) d\mu = \int_\Omega \psi [T(u) f(u) + T(u) d\nu]. \tag{19}$$

Let us choose T with compact support in (a, b). Then, $T(u_n) f(u_n)$ is uniformly bounded and converges $a.e.$ to $T(u) f(u)$: therefore, the convergence holds in $L^1(\Omega)$. Coupled with (19), we deduce that $\int_\Omega \psi T(u) d\nu = 0$, so that, by arbitrarity of ψ, $T(u)|\nu| \equiv 0$. Choosing a sequence of functions $T = T_n$ increasing to $\chi_{(a,b)}$, we deduce that $|\nu|\big([a < u < b]\big) = 0$.

Now, we may write $\nu = \nu_a - \nu_b$ where ν_a, ν_b are respectively carried by $[u = b]$ and $[u = a]$. Assume $b < \infty$. Then, there exists $\beta \in (a, b)$ such that $f \ge 0$ on $[\beta, b)$. Let us choose T with compact support in (β, ∞). Then, $T(u_n) f(u_n) \ge 0$. It follows from (19) and Fatou' Lemma that, for all $\psi \ge 0$

$$\int_\Omega \psi T(u) f(u) \le \liminf_{n\to\infty} \int_\Omega \psi T(u_n) f(u_n) = \int_\Omega \psi [T(u) f(u) + T(u) d\nu_b].$$

We deduce that $\nu_b \ge 0$. Similarly, if $a > -\infty$, we prove $\nu_a \ge 0$.

Step 5: Let $u, w \in H$ satisfying (16). We will prove that

$$\forall U \in D(\overline{A}),\ \forall W \in \overline{A}U,\ \ (u - U, w - W)_H \ge 0. \tag{20}$$

By maximality of $\overline{A}$, it will follow that $u \in D(\overline{A})$ and $w \in \overline{A}u$.

By Step 4, $f(U) \in L^1(\Omega)$ and there exists μ_a, μ_b positive measures carried respectively by $[U = a]$ and $[U = b]$ such that

$$f(U) + \mu_b - \mu_a \in H',\ W = J\big(f(U) + \mu_b - \mu_a\big).$$

Thus

$$(u - U, w - W)_H =< f(u) - f(U) + \nu_b - \nu_a - \mu_b + \mu_a, u - U >_{H' \times H} .$$

For $r \in \mathbb{R}$, let $T(r)$ denote the projection of r onto $[-1, 1]$. Let $\mu = \nu_b - \nu_a - \mu_b + \mu_a$. Then, $T(u - U)\mu$ is a well-defined measure and, if $-\infty < a < b < +\infty$

$$T(u - U)\mu = T(b - U)\nu_b - T(a - U)\nu_a - T(u - b)\nu_b + T(u - a)\nu_a \geq 0,$$

so that $m(u - U) \geq 0\,|\mu|$-a.e. where $\mu = m\,|\mu|$. Since $(f(u) - f(U)(u - U) \geq 0$ a.e. also, we may apply Lemma 7 in Appendix and write

$$(u - U, w - W)_H = \int_\Omega (u - U)\big(f(u) - f(U) + d[\nu_b - \nu_a - \mu_b + \mu_a]\big) \geq 0,$$

whence (20) when a, b are finite. If $a = -\infty$, the proof is the same, but forgetting ν_a, μ_a which are then equal to zero and similarly if $b = +\infty$.

□

Remark 2. If $u \in H$, then the set $[|u| = \infty]$ is of zero capacity. Therefore $\nu_a \equiv 0$ if $a = -\infty$ and $\nu_b \equiv 0$ if $b = +\infty$. In particular, $\nu_b - \nu_a \equiv 0$ when $(a, b) = \mathbb{R}$. Actually, in this case, we can prove directly that $f(u_n)$ converges in $L^1(\Omega)$ to $f(u)$. Indeed, it follows from (18) that $|u_n f(u_n)|$ is bounded in $L^1(\Omega)$. Therefore $(f(u_n))_{n \geq n}$ is uniformly integrable on Ω. Together with the a.e. convergence of $f(u_n)$ to $f(u)$, this implies by Vitali's Lemma that the convergence of $f(u_n)$ holds in $L^1(\Omega)$. Then we can pass to the limit and find directly that $u + Jf(u) = g$.

Remark 3. It follows from the characterization (16) that $\overline{A}$ is indeed a *multivalued* operator in general, even in dimension $N = 1$. For instance, let us choose

$$N = 1,\ \Omega = [0, 1],\ (a, b) = (0, \infty), \delta_\zeta = \text{ Dirac mass at } \zeta = 1/2,$$

$$u \in H,\ \forall x \in [0, 1] \setminus \{\zeta\},\ u(x) > 0,\ u(\zeta) = 0,\ f(u) \in L^1(\Omega).$$

Then, we deduce from (16) that

$$\overline{A}u = \{f(u) - \lambda\delta_\zeta; \lambda \in (0, \infty)\}.$$

Remark 4. We always have

$$\overline{D(\overline{A})} = \overline{D(A)} = D := \{u \in H; a \leq u \leq b \ a.e.\ \}. \tag{21}$$

Indeed, the inclusion $\overline{D(A)} \subset \overline{D(\overline{A})} \subset D$ is obvious. Now, let $u \in D$. Let $(a_n)_{n \geq 0}$ be a sequence which strictly decreases to a and let $(b_n)_{n \geq 0}$ be a sequence which strictly increases to b. Let us define

$$\forall x \in \mathbb{R}^N,\ u_n(x) = \max\{a_n, \min\{u(x), b_n\}\}. \tag{22}$$

Classically, the "truncation" operator is continuous on H so that $u_n \to u$ in H. And , we have $u_n \in D(A)$, since $u_n \in H, f(u_n) \in L^\infty(\Omega) \subset L^2(\Omega)$, whence (21).

2.2. **Global existence.** Now, we may use the theory of maximal monotone operators (see [2]) and the fact that $\overline{A}$ is a subdifferential operator to deduce the following main result of global existence of solutions (recall the definitions of D in (21) and of $\mathcal{M}$ in (14).

Theorem 2. *Let $u_0 \in D$. Then, there exists a unique solution u of the following*

$$\left\{\begin{array}{l} u \in C([0,\infty);H),\ \forall \tau > 0,\ u \in W^{1,\infty}([\tau,\infty);H), \\ f(u) \in L^\infty\big([\tau,\infty);L^1(\Omega)\big),\ f(u)+\nu \in L^\infty\big((\tau,\infty);H' \cap \mathcal{M}\big), \\ \mathbf{a.e.t} \in (\mathbf{0},\infty),\ \mathbf{u_t(t)} - \mathbf{\Delta u_t(t)} + \mathbf{f(u(t))} + \nu(\mathbf{t}) = \mathbf{0}, \\ \nu(t) = \nu_b(t) - \nu_a(t),\ \nu_a(t),\nu_b(t) \geq 0, \\ \nu_a(t)\big([a<u(t)]\big) = 0 = \nu_b(t)\big([u(t)<b]\big), \\ u(0) = u_0. \end{array}\right\} \tag{23}$$

Remark 5. The main equation in (23) is satisfied in H' for $a.e.t$. Actually, u is right-differentiable at each point, $t \to u_t(t^+) \in H$ is right-continuous, the measures $\nu_a(t), \nu_b(t)$ are defined for all t and

$$u_t(t^+) - \Delta u_t(t^+) + f(u(t)) + \nu_b(t) - \nu_a(t) = 0. \tag{24}$$

The measure $\nu(t)$ is singular with respect to the Lebesgue measure and carried by the set $[u(t)=a] \cup [u(t)=b]$ which is of Lebesgue measure zero. This measure is equal to zero if $(a,b) = \mathbb{R}$. But it may indeed be different from zero for some specific values of t, even in dimension $N=1$ as shown by an example below.

Remark 6. Note that it is possible to start with initial data in D. Thus, when a,b are finite, even $u_0 \equiv a$ and $u_0 \equiv b$ are allowed ! But, immediately for $t>0$, the set $[u(t)=a] \cup [u(t)=b]$ is of Lebesgue measure zero.

Proof of Theorem 2. Since $\overline{A} = \partial\Phi$, by well-known abstract results (see e.g. [2]), for all $u_0 \in \overline{D(\overline{A})} = D$, there exists a unique solution of the following evolution problem

$$\left\{\begin{array}{l} \forall \tau \in (0,\infty), u \in C([0,\infty);H) \cap W^{1,\infty}([\tau,\infty);H), \\ \forall t > 0,\ u(t) \in D(\overline{A}),\ a.e.t > 0,\ u_t(t) + \overline{A}u(t) \ni 0 \\ u(0) = u_0. \end{array}\right. \tag{25}$$

Moreover, $t \in (0,\infty) \to u(t)$ is right-differentiable and its right-derivative $u_t(t^+)$ is right-continuous on $(0,\infty)$ and satisfies

$$\forall t > 0, \quad u_t(t^+) + \overline{A}^0 u(t) = 0,\ \ t \in (0,\infty) \to \|\overline{A}^0 u(t)\|_H \textit{ is nonincreasing}, \tag{26}$$

where for $w \in D(\overline{A})$, $\overline{A}^0 w$ is the projection of 0 onto the closed convex set $\overline{A}w$.

By Theorem 1, since $u(t) \in D(\overline{A})$ and $-u_t(t^+) = \overline{A}^0 u(t)$, we have for $t \in [\tau,\infty)$

$$\|f(u(t))\|_{L^1(\Omega)} \leq C[\|u(t)\|_H^2 + \|\overline{A}^0 u(t)\|_H^2 + 1] \leq C[\|u_0\|_H^2 + 1 + \|\overline{A}^0 u(\tau)\|_H^2],$$

where the last inequality uses (26). Moreover, thanks to (16), there exist two positive measures $\nu_a(t), \nu_b(t) \in \mathcal{M}$ such that

$$u_t(t^+) + J(f(u(t) + \nu_b(t) - \nu_a(t)) = 0 \ \ or \ \ u_t(t^+) - \Delta u_t(t^+) + f(u(t)) + \nu_b(t) - \nu_a(t) = 0,$$

$$[\nu_a(t)|\big([u(t)=a]\big) = 0,\ \nu_b(t)\big([u(t)=b]\big) = 0,$$

$$\|f(u(t)) + \nu_b(t) - \nu_a(t)\|_{H'} \leq \|f(u(\tau) + \nu_b(\tau) - \nu_a(\tau)\|_{H'}.$$

This yields the existence results and the properties (23) announced in Theorem 2. For the uniqueness, we just remark that, thanks to Theorem 1 (and (16) in particular), a solution of (23) is also a solution of (25) and use the uniqueness of the solutions of (25).

□

Remark 7. *Measures do appear in the derivatives of the solutions in Theorem 2:* Indeed, let us assume that

$$N = 1, \Omega = (0,1), \ \ (a,b) = (0,\infty), \ f(u) := \ln u, \zeta = 1/2$$

Let us choose $u_0 \in D(A) \cap H^2(\Omega)$ such that

$$\forall x \neq \zeta, u_0(x) > 0, \ u_0(\zeta) = 0, \ln u_0 \in L^2(\Omega), \ J\left(\ln u_0\right)(\zeta) > 0. \tag{27}$$

Then, the corresponding solution of Theorem 2 satisfies

$$u_t(0^+) = -J\left(\ln u_0 - \nu(0)\right), \ \nu(0) = \lambda_m \delta_\zeta, \ \lambda_m = J\left(\ln u_0\right)(\zeta)/J\left(\delta_\zeta\right)(\zeta). \tag{28}$$

In other words, the right-derivative of the solution does involve a positive measure $\nu(0)$.

To build a function u_0 satisfying (27), we first choose

$$v_1 \in H^2(\Omega), v_1(x) = (x-\zeta)^2 \text{ for } x \text{ close to } \zeta, \forall x \neq \zeta, v_1(x) > 0.$$

And we construct an increasing sequence $v_n \in H^2(\Omega), n \geq 2$ equal to v_1 on $(\zeta - 1/n, \zeta + 1/n)$ and such that $\forall x \neq \zeta, \lim_{n\to\infty} v_n(x) = 2$. Note that $\ln v_1 \in L^p(\Omega)$ for all $p \in (0,1)$ and, by monotonicity, $\ln v_n$ converges in $L^p(\Omega)$ to $\ln 2 > 0$. Since $N = 1$, this convergence holds also in H' and, consequently, $J\left(v_n\right)$ converges uniformly to $J(2) = 2 > 0$. To satisfy (27), we then choose $u_0 := v_n$ with n large enough.

Next, by the proof of Theorem 2, we know that $u_t(0^+) = -\overline{A}^0 u_0$, and thanks to the Remark 3, $\overline{A}^0 u_0 = \ln u_0 - \lambda_m \delta_\zeta$ where λ_m minimizes $\lambda \to h(\lambda)$ where

$$h(\lambda) := \|J(\ln u_0 - \lambda\delta_\zeta)\|_H^2 = \|J(\ln u_0)\|_H^2 - 2\lambda J(\ln u_0)(\zeta) + \lambda^2 J(\delta_\zeta)(\zeta).$$

Since $J(\ln u_0)(\zeta) > 0$, the minimum is reached for $\lambda = J(\ln u_0)(\zeta)/J(\delta_\zeta)(\zeta)$, whence (28).

We do not know whether $\nu(t)$ is zero or not for $t > 0$. But, it is clear that one cannot have

$$\forall t \in (0,\tau), u(t,\zeta) = 0, \forall x \neq \zeta, u(t,x) > 0,$$

for some $\tau > 0$. Indeed, by (16), we would have $\nu(t) = \lambda(t)\delta_\zeta, \lambda(t) \geq 0$. Next, for all $t \in (0,\tau)$

$$u(t) - u(t)_{xx} + \int_0^t [\ln u(s) - \lambda(s)\delta_\zeta] ds = u_0 - (u_0)_{xx}. \tag{29}$$

In particular $u_x(t,0^+)(\geq 0), u_x(t,0^-)(\leq 0)$ exist and, by integration in space from $\zeta - \eta$ to $\zeta + \eta$, and after letting $\eta \to 0$, we obtain

$$0 \geq -u_x(t,\zeta^+) + u_x(t,\zeta^-) = \int_0^t \lambda(s) ds.$$

Since $\lambda(\cdot)$ is nonnegative, this implies $\lambda(t) = 0$ a.e.t and $u_x(t,\zeta)$ exists and is equal to 0.

Going back to (29), since $\int_0^t \ln u(s,x) ds$ tends to $-\infty$ as $x \to \zeta$, we have $u(t)_{xx} < 0$ in a neighborhood of $x = \zeta$. This is not compatible with $u_x(t,\zeta) = 0, u(t,x) > 0$ for $x > \zeta$.

Conclusion: It follows from this analysis that, either $u(t) > 0$ for $t > 0$ or $u(t)$ vanishes at some points $x \neq 0$. According to the numerical computations in Subsection 2.4, it is very likely that $u(t)$ vanishes at (exactly) two points which move on a symmetric curve around ζ.

2.3. **Asymptotic behaviour.** Let $S_t u_0 := u(t)$, where u is the solution obtained in Theorem 2. By the theory of maximal monotone operators [2], $\{S_t\}_{t\geq 0}$ is a contraction semigroup on $\overline{D(\bar{A})}$ for the $\|\cdot\|_H$ norm. Since $\bar{A} = \partial\Phi$ is a subdifferential, we also know that $S_t u_0 \in D(\bar{A})$ for all $t > 0$ and for all $u_0 \in \overline{D(\bar{A})}$.

In this Subsection, we focus on the asymptotic behaviour of the dynamical system $\{S_t\}_{t\geq 0}$ [4]. By definition, a *stationary solution* of this dynamical system is a function $u_0 \in \overline{D(\bar{A})}$ such that $S_t u_0 = u_0$ for all $t \geq 0$. By the regularizing effect recalled above, any such stationary solution u_0 belongs to $\bar{A}$, and by derivation, $0 \in \bar{A}u_0$. Thus,

$$\mathcal{S} := \{v \in D(\bar{A}) \ : \ 0 \in \bar{A}v\}$$

is the set of stationary solutions. By maximality of $\overline{A}$, it is a closed set of H. Let

$$Z_f := \{c \in (a,b) \ : \ f(c) = 0\}$$

denote the set of roots of f.

Proposition 1. *Any stationary solution $u_\star \in \mathcal{S}$ satisfies $u_\star \in Z_f$ a.e. in Ω. In particular, if $Z_f = \emptyset$, the dynamical system $\{S_t\}_{t\geq 0}$ has no stationary solution.*

Proof. Let $u_\star \in \mathcal{S}$. Then, $u_\star \in D(\overline{A}), 0 \in \overline{A}(u_\star)$. By (16) in Theorem 1, $f(u_\star) \in L^1(\Omega)$ and there exists a measure $\nu \in \mathcal{M}$ which is singular with respect to the Lebesgue measure and such that $0 = J(f(u_\star) - \nu)$. This means $0 = f(u_\star) - \nu$ which implies that $\nu \equiv 0, f(u_\star) = 0$ a.e., whence Proposition 1. $\square$

Up to the end of this Subsection, we will assume that f has a root in (a,b). Without loss of generality, we may assume that

$$(30) \qquad 0 \in (a,b) \quad \text{and} \quad f(0) = 0.$$

As a consequence, $F(s) := \int_0^s f(\sigma)d\sigma \geq 0$ for all $s \in (a,b)$.

Using Lasalle's invariance principle [4], we will prove that under additional natural assumptions, any solution of (5) converges to a stationary solution. For this purpose, we introduce for every $u_0 \in \overline{D(\bar{A})}$ its ω-limit set defined by

$$\omega(u_0) := \{v \in H \ : \ \exists t_n \to +\infty \text{ such that } S_{t_n}u_0 \to v \text{ in } H \text{ as } n \to +\infty\}.$$

We first derive global a priori estimates. As a shortcut, we denote H^2 the space $H^2_{per}(\Omega)$.

Lemma 1. *Assume that f satisfies* (30)*. Let $u_0 \in H$ such that $\Phi(u_0) < +\infty$ and let $u \in W^{1,2}_{loc}([0,+\infty);H)$ denote the unique solution of*

$$(31) \qquad u_t + \bar{A}u \ni 0, \quad u(0) = u_0.$$

Then u is bounded in H and

$$(32) \qquad \Phi(u(t)) + \int_s^t \|u_t(\sigma)\|_H^2 \, d\sigma = \Phi(u(s)) \quad \textit{for all } 0 \leq s \leq t < +\infty.$$

Moreover, if $u_0 \in H^2$, then u is bounded in H^2.

Proof. The identity (32) is standard for equations governed by subdifferentials (see [2], Proposition 3.1). It uses the property that $\frac{d}{dt}\Phi(u(t)) = (\overline{A}^0 u(t), u(t))_H$ a.e. t. Thus, we multiply $u_t(t) + \overline{A}^0 u(t) = 0$ by $u_t(t)$ for the scalar product in H and we integrate in time to obtain (32)

Multiplying $u_t(t) - \Delta u_t(t) + f(u(t)) + \nu_b(t) - \nu_a(t) = 0$ by $u(t)$ provides the bound on $u(t)$ in $H:$ indeed, we then use $u(t)f(u(t)) \geq 0$, since $f(0) = 0$ and

$$u(t)(\nu_b(t) - \nu_a(t)) = b\nu_b(t) - a\nu_a(t) \geq 0 \text{ since } 0 \in (a,b).$$

We deduce

$$0 \geq \int_\Omega u(t)u_t(t) + \nabla u(t)\nabla u_t(t) = \frac{d}{dt}\frac{1}{2}\int_\Omega u^2(t) + |\nabla u(t)|^2.$$

This yields $\|u(t)\|_H \leq \|u_0\|_H$.

For the H^2-estimate, we may use $u(t) = \lim_{n\to\infty}(I + \frac{t}{n}\overline{A})^{-n}u_0$ in H (see [2]) and iterate the estimate

$$\forall \lambda > 0, \forall w \in H \cap H^2,\ \|(I-\Delta)(I+\lambda\overline{A})^{-1}w\|_{L^2(\Omega)} \leq \|(I-\Delta)w\|_{L^2(\Omega)} \tag{33}$$

which can be proved as follows: let $v := (I + \overline{A})^{-1}w$. By (17) and the Appendix, we know that $w \in H \cap H^2$ implies $v \in H^2, f(v) \in L^2$ and we have

$$v - \Delta v + \lambda f(v) = w - \Delta w.$$

We multiply this equation by $v - \Delta v$ and using $\int_\Omega (v - \Delta v)f(v) \geq 0$, we deduce (33). It follows that

$$\|(I-\Delta)u(t)\|_{L^2} \leq \lim_{n\to\infty} \|(I-\Delta)(I + \frac{t}{n}\overline{A})^{-n}u_0\|_{L^2} \leq \|(I-\Delta)u_0\|_{L^2}.$$

□

Remark 8. We deduce from the H^2-estimate that, if $u_0 \in H \cap H^2$, then $\int_0^t \nu_a(s)ds$ and $\int_0^t \nu_b(s)ds$ are functions which are at least in $L^\infty([0,T] : L^1(\Omega))$ since we have:

$$u(t) - \Delta u(t) + \int_0^t [f(u(s)) + \nu_b(s) - \nu_a(s)]ds = u_0 - \Delta u_0.$$

We always have [2]:

$$D(\bar{A}) = D(\partial\Phi) \subset D(\Phi) \subset \overline{D(\bar{A})}.$$

For $v \in H$ and $\mathcal{B} \subset H$, we denote $d(v, \mathcal{B})$ the distance of v to $\mathcal{B}$, defined by

$$d(v, \mathcal{B}) := \inf_{w\in\mathcal{B}} \|v - w\|_H .$$

Theorem 3. *Assume that $D(\Phi) = \overline{D(\bar{A})}$, that Φ is continuous on $D(\Phi)$ and that f satisfies* (30)*. For all $u_0 \in D(\Phi) \cap H^2$, $\omega(u_0)$ is a nonempty compact connected subset of $\mathcal{S}$ and $d(S_t u_0, \omega(u_0)) \to 0$ as $t \to +\infty$.*

Proof. By Lemma 1, the set $\cup_{t\geq 0}\{S_t u_0\}$ is relatively compact in H (recall that the injection $H^2 \subset H$ is compact). By (32), for all $u_0 \in D(\Phi)$, the function $t \mapsto \Phi(S_t u_0)$ is nonincreasing; moreover, if $u_0 \in D(\Phi)$ satisfies $\Phi(S_t u_0) = \Phi(u_0)$ for all $t \geq 0$, then $(S_t u_0)_t \equiv 0$ for a.e. $t \geq 0$ and so $u_0 \in \mathcal{S}$. In other words, Φ is a strict Lyapunov function for the dynamical system $\{S_t\}_{t\geq 0}$ on $D(\Phi)$. The conclusion follows from a standard result (see [4] for details). □

Example 1. If a and b are finite and if F is continuous on $[a, b]$, then

$$D(\Phi) = \overline{D(\bar{A})} = \{v \in H\ :\ v \in [a, b] \text{ a.e. in } \Omega\},$$

and Φ is continuous on $D(\Phi)$, by Lebesgue's dominated convergence theorem. This situation applies in particular to the standard choice of $f = F'$ given by (2).

Example 2. If $(a, b) = \mathbb{R}$ and if there exists a constant $C > 0$ such that

$$|F(s)| \leq C(1 + |s|^p) \quad \forall s \in \mathbb{R}, \tag{34}$$

where $p = 2^\star = 2N/(N-2)$ if $N \geq 3$, p is any real if $N = 2$ and no growth restriction is required if $N = 1$, then $D(\Phi) = H$ and Φ is continuous on H. For instance, F can be a convex polynomial with a critical or subcritical growth.

Example 3. If a is finite, if $b = +\infty$, if F is continuous at a and if F has at most a critical growth near $b = +\infty$, then

$$D(\Phi) = \{v \in H \ : \ v \geq a\} = \overline{D(\bar{A})}$$

and F is continuous on $D(\Phi)$. For instance, the standard logarithmic function $f(s) = \ln(1+s)$ on $(-1, +\infty)$ has an antiderivative F which satifies such assumptions in any dimension N.

As a consequence of Theorem 3, we have

Corollary 1. *Assume that the assumptions of Theorem 3 are satisfied. If* 0 *is the unique root of* f *in* (a, b)*, then for all* $u_0 \in D(\Phi)$*,* $S_t u_0$ *converges to* 0 *in* H *as* $t \to +\infty$*.*

Proof. If 0 is the unique root of f, then $\mathcal{S} = \{0\}$ by Proposition 1. If $u_0 \in D(\Phi) \cap H^2$, Theorem 3 implies that $S_t u_0 \to 0$ in H as $t \to +\infty$. If $u_0 \in D(\Phi)$, then for every $\varepsilon > 0$, we can find $u_0^\varepsilon \in D(\Phi) \cap H^2$ such that $\|u_0 - u_0^\varepsilon\|_{H,\Gamma} \leq \varepsilon$. By monotonicity,

$$\|S_t u_0 - S_t u_0^\varepsilon\|_H \leq \|u_0 - u_0^\varepsilon\|_H \quad \forall t \geq 0.$$

On the other hand, $S_t u_0^\varepsilon \to 0$ as $t \to +\infty$, so that for t large enough, $\|S_t u_0\|_H \leq 2\varepsilon$. This proves the assertion. □

The result above implies that if $N = 1$, $S_t u_0 \to 0$ uniformly. In particular, $\bar{A} S_t u_0 = A S_t u_0$ for t large enough. More generally, we have:

Corollary 2. *Assume that the assumptions of Theorem 3 are satisfied. If* $N = 1$ *and if* $u_0 \in D(\Phi)$*, then for* t *large enough,* $S_t u_0$ *takes values in a compact subset of* (a, b)*. If* $N = 2$ *or* 3 *and if* $u_0 \in D(\Phi) \cap H^2$*, then for* t *large enough,* $S_t u_0$ *takes values in a compact subset of* (a, b) *a.e. in* Ω*. In theses cases, for* t *large enough,* $f(S_t u_0) \in L^\infty(\Omega)$ *and* $\bar{A} S_t u_0 = A S_t u_0$*.*

Proof. By Theorem 3, for every $u_0 \in D(\Phi) \cap H^2$, $d(S_t u_0, \omega(u_0)) \to 0$ as $t \to +\infty$ and $\omega(u_0) \subset \mathcal{S}$. By Proposition 1, for every $v \in \omega(u_0)$, $v(x) \in Z_f$ a.e.$x \in \Omega$. Since $t \to S_t u_0$ is bounded in H^2, if $N \leq 3$, $t \to S_t u_0$ is bounded in $L^\infty(\Omega)$. Thus, if $v \in \omega(u_0)$, it takes its values in a bounded set of Z_f and, since f is maximal monotone, this set is necessarily included in a compact set Z of (a, b), say $Z \subset (a', b')$ where $a', b' \in (a, b)$.

If $N = 1$ and if $u_0 \in D(\Phi)$, then by arguing as in the proof of Corollary 1, we have $d(S_t u_0, \mathcal{S}) \to 0$. Taking advantage of the continuous injection $H \subset C_{per}(\overline{\Omega})$, we conclude that for t large enough, $S_t u_0$ takes values in (a', b').

Assume now that $N = 2$ or $N = 3$ and that $u_0 \in D(\Phi) \cap H^2$. Let us prove that $d_\infty(S_t u_0, \omega(u_0)) \to 0$ as $t \to \infty$ where d_∞ is the uniform distance. If not, there exists $t_n \to \infty$ and $\varepsilon > 0$ such that $d_\infty(S_{t_n} u_0, \omega(u_0)) \geq \varepsilon$. But, up to a subsequence, $S_{t_n} u_0$ converges weakly in H^2 to some $u^\star$. By compactness embedding of H^2, the convergence holds also at the same time in H and uniformly. Thus, we have $u^\star \in \omega(u_0)$ and $d_\infty(S_{t_n} u_0, \omega(u_0)) \leq \|S_{t_n} u_0 - u^\star\|_H \to 0$, whence a contradiction. Thus, we do have $d_\infty(S_t u_0, \omega(u_0)) \to 0$ as $t \to \infty$ so that $S_t u_0$ takes it values in (a', b') for t large enough and Corollary 2 follows. □

2.4. **Numerical approach and results.** In this section, we present numerical results which illustrate the theoretical results.

2.4.1. *The choice of the data.* We consider equation (5) with the nonlinearity $f(s) = \ln s$ defined on $(a, b) = (0, +\infty)$; the domain is the segment $\Omega = (0, 1)$. We choose an initial condition $u_0 \in H^2 \cap C^1_{per}([0, 1]; \mathbb{R})$ such that a measure ν with positive total mass may appear in the derivative at time $t \geq 0$, where ν is defined by (6).

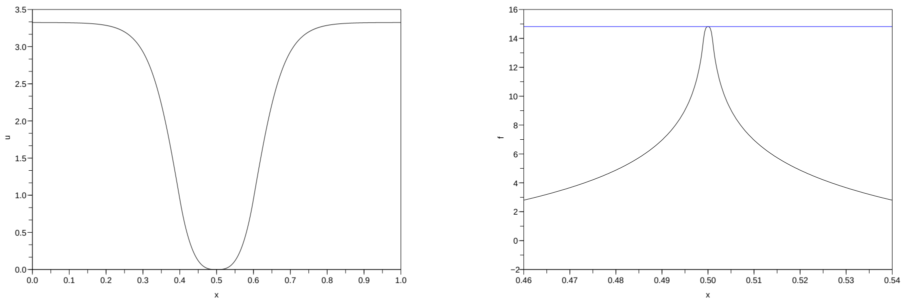

FIGURE 1. u_0 (left) and $-f_\varepsilon(u_0)$ (right)

This function u_0 is defined by (see Figure 1, left)

$$u_0(x) = \begin{cases} g(x-0.5) & \text{if } |x-0.5| \le L \\ 95B\tanh(\delta(x-0.5-L)/B) + g(L) & \text{if } x \in [0.5+L, 1] \\ -95B\tanh(\delta(x-0.5+L)/B) + g(L) & \text{if } x \in [0, 0.5-L], \end{cases}$$

where $g(s) = 950s^3$. The coefficients are $L = 0.1$, $\delta = 30L^2$ and $B = 0.025$. For every $x \in [0,1]$, $u_0(x) \ge 0$ with equality if and only if $x = 0.5$.

2.4.2. *A regularized approximation.* Because of the singularity, we cannot compute the solution directly and we compute instead (an approximation of) a regularized solution. The nonlinearity f is regularized by

$$f_\varepsilon(s) = \begin{cases} \ln(s) & \text{if } s \ge \varepsilon \\ s/\varepsilon + \ln(\varepsilon) - 1 & \text{if } s \le \varepsilon, \end{cases}$$

so that $f_\varepsilon \in C^1(\mathbb{R},\mathbb{R})$ is nondecreasing on $\mathbb{R}$ and globally Lipschitz continuous. Figure 1 (right) represents the graph of $-f_\varepsilon(u_0)$ on the interval $[0.46, 0.54]$; the horizontal (blue) line is the value of $-f_\varepsilon(0)$.

We denote $u^\varepsilon \in C^1([0,+\infty); H)$ the solution to

$$u_t^\varepsilon + Jf_\varepsilon(u^\varepsilon) = 0 \quad \forall t \ge 0, \quad u^\varepsilon(0) = u_0. \tag{35}$$

This solution is uniquely defined by application of the Cauchy-Lipschitz theorem on the Banach space H. Problem (35) is equivalent to

$$u_t^\varepsilon - \Delta u_t^\varepsilon + f_\varepsilon(u^\varepsilon) = 0 \quad \forall t \ge 0, \quad u^\varepsilon(0) = u_0. \tag{36}$$

We have

Proposition 2. *For every $T > 0$, u^ε converges strongly in $C([0,T];H)$ to the solution u of (23) as $\varepsilon \to 0$. Moreover, $f_\varepsilon(u^\varepsilon)$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and converges in the sense of measures to $f(u(t,\cdot)) + \nu(t)$ defined by (6).*

Proof. If we denote by A^ε the maximal monotone operator associated with Jf_ε, with the same proof as we did in the Appendix for the Yosida approximation of f, we may check that, for all $g \in H \cap H^2$ and all $\lambda > 0$, $(I+\lambda A^\varepsilon)^{-1}g \to (I+\lambda\overline{A})^{-1}g$ as $\varepsilon \to 0$. By density, this holds for all $g \in H$. By a standard result (see [2, Théorème 3.16]), it implies that u^ε converges strongly in $C([0,T];H)$ for all $T > 0$ to the solution u of (5) as $\varepsilon \to 0$.

By (36), $f_\varepsilon(u^\varepsilon) = -(I-\Delta)(u_t^\varepsilon)$ converges in the sense of distributions on $(0,T)\times\Omega$ to $-(I-\Delta)u_t$ which is equal to $f(u(t)) + \nu(t)$ as defined in (6). Moreover, as in the proof of Proposition 3, we obtain that $f_\varepsilon(u^\varepsilon(t))$ is bounded in $L^1(\Omega)$ uniformly in time. Whence the convergence in the sense of measures.

□

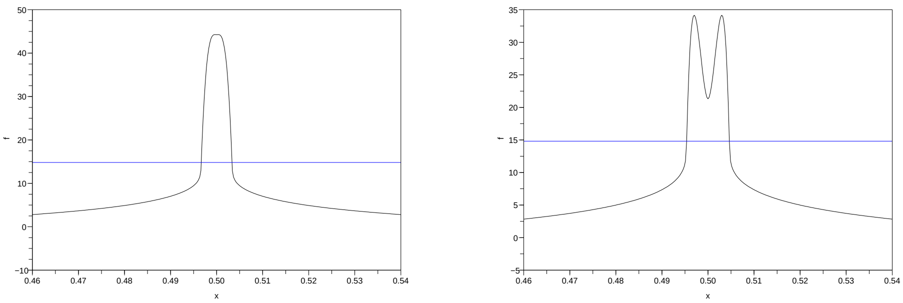

FIGURE 2. $-f_\varepsilon(u(t,\cdot))$ at $t = 0.025$ (left) and $t = 0.25$ (right)

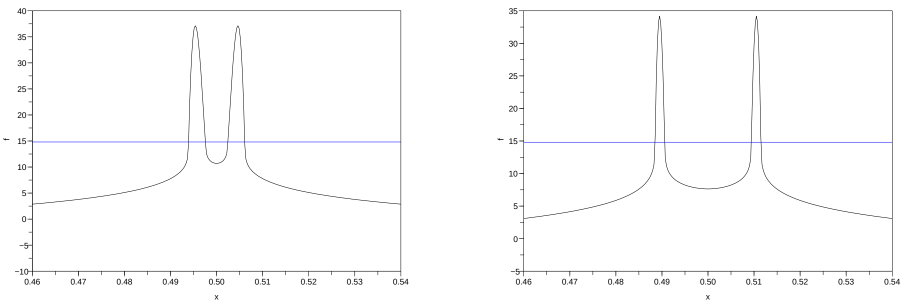

FIGURE 3. $-f_\varepsilon(u(t,\cdot))$ at $t = 0.5$ (left) and $t = 2$ (right)

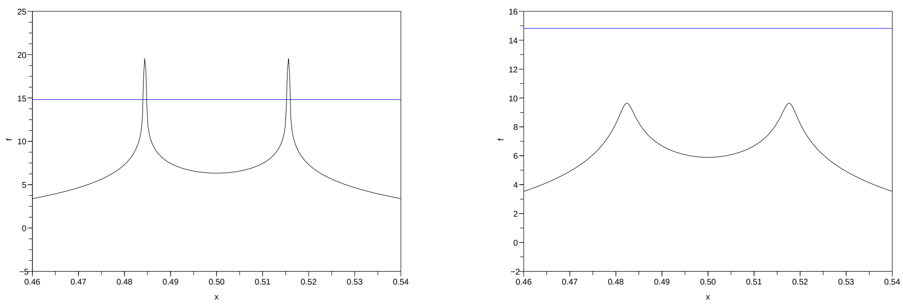

FIGURE 4. $-f_\varepsilon(u(t,\cdot))$ at $t = 4$ (left) and $t = 5$ (right)

2.4.3. *Space and time discretization of the regularized problem.* For the numerical approximation of (36), we use a continuous piecewise linear (P^1 conforming) finite element approximation for the space discretization and a backward Euler scheme for the time discretization. We use a uniform subdivision $x_i = ih$ $(0 \le i \le M)$ of

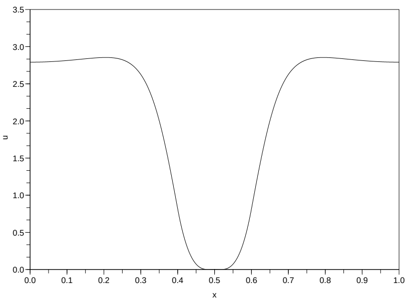

FIGURE 5. $u(t,\cdot)$ at $t = 5$

$[0,1]$, where $= 1/M$ denotes the mesh step ($M \in \mathbb{N}^\star$). The P^1 finite element space is

$$H_h := \left\{ v_h \in C_{per}([0,1],\mathbb{R}) \ : \ (v_h)_{|[x_i,x_{i+1}]} \text{ is affine for } i = 0, 1, \dots, M-1 \right\},$$

where $C_{per}([0,1]) = \{v \in C([0,1],\mathbb{R}) \ : \ v(0) = v(1)\}$. Let $\delta t > 0$ denote the time stepsize. The approximated problem reads : let $u_{0,h} = u^\varepsilon_{h,0} \in H_h$ denote the P^1 interpolate of u_0 and for $n = 0,\ 1,\ 2,\ \dots$, let $u^\varepsilon_{h,n+1} \in H_h$ solve

$$\int_\Omega (u^\varepsilon_{h,n+1} - u^\varepsilon_{h,n}) v_h + \int_\Omega \nabla(u^\varepsilon_{h,n+1} - u^\varepsilon_{h,n}) \cdot \nabla v_h + \delta t \int_\Omega f_\varepsilon(u^\varepsilon_{h,n+1}) v_h = 0 \qquad (37)$$

for all $v_h \in H_h$. By monotonicity, for a given $u_{0,h} \in H_h$ the sequence $(u^\varepsilon_{h,n})_{n\geq 0}$ is uniquely defined. Let $u^\varepsilon_{h,\delta t} \in C([0,+\infty);H)$ denote the continuous piecewise linear interpolate associated to this sequence. By standard methods (see [14]), we check that: *For all $T > 0$, $u^\varepsilon_{h,\delta t}$ converges weakly in $W^{1,2}(0,T;H)$ and uniformly in $\overline{\Omega} \times [0,T]$ to u^ε, as $h \to 0$ and $\delta t \to 0$.*

2.4.4. *Numerical results.* We have computed with a `Scilab`[1] program the sequence $(u^\varepsilon_{h,n})_n$ defined by (37) for various values of ε, h and δt. The nonlinear system is solved at every time step by a Newton-like algorithm.

Figures 2-5 represent the graph of $-f_\varepsilon(u^\varepsilon_{h,\delta t})$ at various times, for $\varepsilon = 1e-6$, $h = 1/4096$ and $\delta t = 0.025$; the horizontal (blue) line is the value of $-f_\varepsilon(0)$. By symmetry of equation (36) and of the initial condition with respect to the map $v(0.5+\cdot) \mapsto v(0.5-\cdot)$ on $[0,1]$, $u^\varepsilon(t)$ is even with respect to $x = 0.5$ at every time t; this symmetry is reproduced at the discrete level, up to computational accuracy.

At the first time step $t = 0.025$, the function

$$w^\varepsilon_{h,\delta t}(\cdot,t) := f_\varepsilon(u^\varepsilon_{h,\delta t}(\cdot,t)) 1_{\{x\in[0,1]\ :\ u^\varepsilon_{h,\delta t}<0\}}$$

looks like the approximation of a (negative) delta measure at $x = 0.5$. For larger times ($t \in [0.5, 4.3]$) , $w^\varepsilon_{h,\delta t}$ looks like the approximation of two (negative) delta mesures at points $0.5 + \bar{x}(t)$ and $0.5 - \bar{x}(t)$, where $\bar{x}$ is an increasing function of t.

In order to support this claim, we have represented the set

$$K^\varepsilon_{h,\delta t} := \{(x,t) \in [0,1] \times [0,5] \ : \ u^\varepsilon_{h,\delta t}(x,t) < 0\},$$

for $h = 1/4096$, $\delta t = 0.0025$ and for decreasing values of ε, namely $\varepsilon = 1e-5$, $\varepsilon = 1e-6$ and $\varepsilon = 1e-7$ (Figures 8 and 9, which were drawn by a `Matlab`[2] graph). For a fixed $t > 0$, the family of sets $K^\varepsilon_{h,\delta t}$ seems to converge, as $\varepsilon \to 0$, to a reunion of two curves symmetric around the $(0.5, t)$ axis, and which could be

[1]`Scilab` is freely available at http://www.scilab.org/

[2]http://www.mathworks.fr/

written $0.5+\bar{x}(t)$ and $0.5-\bar{x}(t)$. Notice that for a fixed $\varepsilon>0$, the set $K^{\varepsilon}_{h,\delta t}$ is stable with respect to refinement in h and δt. This is shown in Tables 6 and 7 which represent the upper-right extremal point of $K^{\varepsilon}_{h,\delta t}$, for various values of the parameters (the upper-left extremal point of $K^{\varepsilon}_{h,\delta t}$ — which is not represented here — is the symmetric of the upper-right extremal point around the $(0.5,t)$ axis, up to computational accuracy).

$M=1/h$	δt	$\varepsilon=1e-5$	$\varepsilon=1e-6$	$\varepsilon=1e-7$
1024.	0.1	4.4	4.4	4.2
2048.	0.05	4.4	4.3	4.3
4096.	0.025	4.4	4.325	4.35
8192.	0.0125	4.4125	4.325	4.3125
16384.	0.00625	4.4125	4.33125	4.325

FIGURE 6. Time-coordinate of the upper-right extremal point of $K^{\varepsilon}_{h,\delta t}$

$M=1/h$	δt	$\varepsilon=1e-5$	$\varepsilon=1e-6$	$\varepsilon=1e-7$
1024.	0.1	0.5166016	0.5166016	0.515625
2048.	0.05	0.5166016	0.5161133	0.5161133
4096.	0.025	0.5166016	0.5163574	0.5163574
8192.	0.0125	0.5164795	0.5163574	0.5162354
16384.	0.00625	0.5165405	0.5162964	0.5162964

FIGURE 7. Space-coordinate of the upper-right extremal point of $K^{\varepsilon}_{h,\delta t}$

We have also computed the total mass

$$M^{\varepsilon}_{h,\delta t}(t):=\int_0^1 f_\varepsilon(u^{\varepsilon}_{h,\delta t}(x,t))1_{\{x\in[0,1]\,:\,u^{\varepsilon}_{h,\delta t}(t,x))<0\}}dx$$

at times $t=1$ and $t=2$, for various values of the parameters (Table 10 and Table 11). Since f_ε is affine for negative real numbers, the function $w^{\varepsilon}_{h,\delta t}(\cdot,t)$ is a continuous piecewise linear function and $M^{\varepsilon}_{h,\delta t}(t)$ can be easily computed by a trapezoidal rule. For a fixed ε and for a fixed t, the value of $M^{\varepsilon}_{h,\delta t}(t)$ is seen to converge as the space and time stepmesh are refined, in agreement with our above analysis; and as ε decreases to 0, $|M^{\varepsilon}_{h,\delta t}(t)|$ increases slightly. All these results are in agreement with the conjecture that $f_\varepsilon(u^\varepsilon(\cdot,t))1_{\{x\in[0,1]\,:\,u^\varepsilon(x,t)<0\}}$ converges to a sum of two delta measures with negative mass, as $\varepsilon\to 0$ (see Proposition 2).

3. The general case.

In this section, we study the full equation (1), with even a more general diffusion, namely

$$u_t+V\cdot\nabla u_t-\mathrm{div}(\Gamma\nabla u_t)+f(u)-\theta u-\alpha\Delta u=\gamma\quad x\in\Omega,\ t>0,\ u(0,\cdot)=u_0, \tag{38}$$

where $V\in\mathbb{R}^N$, Γ is a symmetric positive definite matrix of size N, α and θ are nonnegative real numbers, and $\gamma:\Omega\times\mathbb{R}_+\to\mathbb{R}$ is a function (which represents a source term). As previously, $f:(a,b)\to\mathbb{R}$ is a continuous nondecreasing function which is maximal monotone (see (10)). We prove global existence of solution and we study the asymptotic behavior.

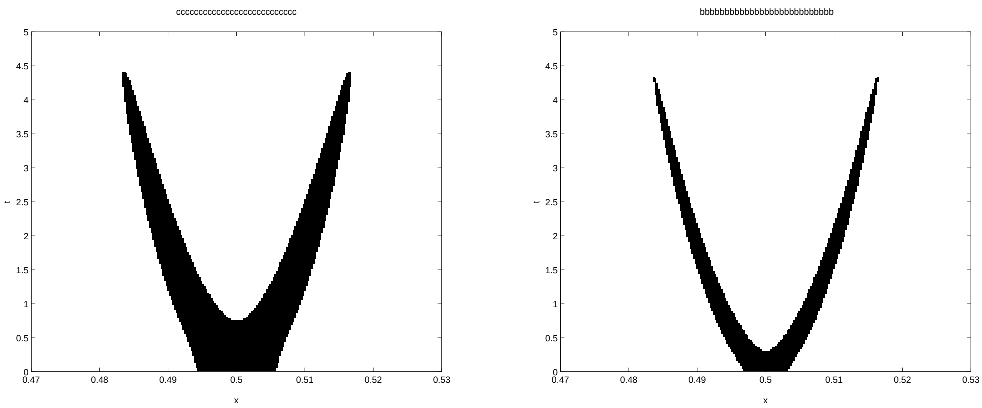

FIGURE 8. Set $K^{\varepsilon}_{h,\delta t}$

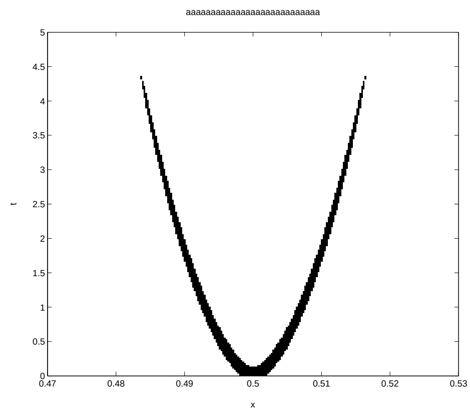

FIGURE 9. Set $K^{\varepsilon}_{h,\delta t}$

$M = 1/h$	δt	$\varepsilon = 1e-5$	$\varepsilon = 1e-6$	$\varepsilon = 1e-7$
1024.	0.1	- 0.0763436	- 0.0980430	- 0.1267127
2048.	0.05	- 0.0750051	- 0.0857377	- 0.1035929
4096.	0.025	- 0.0750933	- 0.0853777	- 0.0957055
8192.	0.0125	- 0.0751139	- 0.0852910	- 0.0923459
16384.	0.00625	- 0.0751336	- 0.0852362	- 0.0924410

FIGURE 10. Total negative mass $M^{\varepsilon}_{h,\delta t}(t)$ at time $t = 1$

3.1. **The general operator $\overline{A}$.** Following the previous section, we work in the Hilbert space $H = H^1_{per}(\Omega)$ equipped with Hilbert norm $\|\cdot\|_H$. We introduce the linear operator

$$Lu = u + V \cdot \nabla u - \operatorname{div}(\Gamma \nabla u)$$

with periodic boundary conditions. By the Lax-Milgram theorem, L is an isomorphism from H onto H' associated to the continuous and coercive bilinear form on $H \times H$ given by

$$(u, v)_{V,\Gamma} := \int_\Omega uv + (V \cdot \nabla u)v + (\Gamma \nabla u) \cdot \nabla v \, dx \quad \forall (u, v) \in H \times H.$$

$M = 1/h$	δt	$\varepsilon = 1e-5$	$\varepsilon = 1e-6$	$\varepsilon = 1e-7$
1024.	0.1	- 0.0487227	- 0.0673515	- 0.1111719
2048.	0.05	- 0.0475348	- 0.0567351	- 0.0805211
4096.	0.025	- 0.0470579	- 0.0539190	- 0.0609115
8192.	0.0125	- 0.0470242	- 0.0535306	- 0.0598122
16384.	0.00625	- 0.0470200	- 0.0534997	- 0.0587499

FIGURE 11. Total negative mass $M^{\varepsilon}_{h,\delta t}(t)$ at time $t = 2$

By definition,

(39) $\langle Lu, v\rangle_{H',H} = (u, v)_{V,\Gamma} \quad \forall (u, v) \in H \times H.$

When $V = 0$, the bilinear form $(\cdot,\cdot)_{V,\Gamma}$ defines a scalar product on H, and we denote

$$\|u\|_{H,\Gamma} := (u, u)^{1/2}_{0,\Gamma} = \int_\Omega u^2 + (\Gamma\nabla u)\cdot\nabla u\,dx \quad (u \in H)$$

the associated norm, which is equivalent to the norm $\|\cdot\|_H$ on H. The dual norm on H' is denoted

(40) $\|v\|_{H',\Gamma} := \sup_{u\in H,\ \|u\|_{H,\Gamma}\le 1} \langle v, u\rangle_{H',H} \quad \text{and also : } |u|_0^2 = \int_\Omega u^2.$

We note that for all $u \in H$, we have

(41) $\int_\Omega (V\cdot\nabla u)u\,dx = 0 \quad \text{and} \quad (u, u)_{V,\Gamma} = (u, u)_{0,\Gamma}.$

Let $J = L^{-1}$. We define the nonlinear operator A on H by

$$\begin{cases} D(A) = \left\{u \in H,\ u \in (a, b) \text{ a.e. and } f(u) \in L^2(\Omega)\right\}, \\ \forall u \in D(A),\ Au = J(f(u)). \end{cases}$$

The closure of A in $H \times H$ is denoted $\overline{A}$. By definition, $(u, v) \in \overline{A}$ if and only if there exists a sequence $(u_n)_n$ in $D(A)$ such that $(u_n, Au_n) \to (u, v)$ in $H \times H$. The domain of $\bar{A}$ is defined as usually by

$$D(\bar{A}) = \{u \in H \ :\ \exists v \in H \text{ such that } (u, v) \in \bar{A}\}.$$

As a shortcut, we denote H^2 the space $H^2_{per}(\Omega)$, and L^2 the space $L^2(\Omega)$. The L^2-norm is denoted $|\cdot|_0$. By standard elliptic regularity [8], the operator L acts isometrically from H^2 onto L^2, and Δ is bounded from H^2 into L^2.

By definition, the operator A has the following monotonicity-like property : for all $(u, \hat{u}) \in D(A) \times D(A)$,

(42) $(Au - A\hat{u}, u - \hat{u})_{V,\Gamma} = \int_\Omega (f(u) - f(\hat{u})(u - \hat{u})dx \ge 0.$

This relation implies:

Proposition 3. *If $u \in D(\bar{A})$, then $u \in (a, b)$ a.e. and $f(u) \in L^1(\Omega)$. Moreover, there exists a constant C such that for all $(u, v) \in \bar{A}$,*

(43) $\int_\Omega |f(u)|dx \le C\left(\|v\|_H + 1\right)\left(\|u\|_H + 1\right).$

Proof. Let $c \in (a, b)$. By continuity, we can find $\eta > 0$ such that $|f(r)-f(c)| \le 1$ for all $r \in \mathbb{R}$ such that $|r-c| \le \eta$. Let $u \in D(A)$. We have $|f(u)| \le |f(c)|+|f(u)-f(c)|$, and by integration on Ω,

$$\int_\Omega |f(u)|dx \quad \le \quad |f(c)||\Omega| + \int_\omega 1dx + \eta^{-1}\int_{\Omega\setminus\omega} (f(u) - f(c))(u - c)dx,$$

where $|\Omega|$ is the Lebesgue measure of Ω and where $\omega = \{x \in \Omega \ : \ |u(x) - c| \le \eta\}$. Using (42) with $\hat{u} = c$, we find

$$\int_\Omega |f(u)|dx \le (|f(c)| + 1)|\Omega| + \eta^{-1} C(|V|, |\Gamma|) \left\|Au - Ac\right\|_H \left\|u - c\right\|_H .$$

Notice that $Lf(c) = f(c)$ so that $Ac = f(c)$. Thus,

$$\int_\Omega |f(u)|dx \le C(\|Au\|_H + 1)(\|u\|_U + 1),$$

for a constant C which depends only on c, $f(c)$, η, $|\Omega|$, $|V|$ and $|\Gamma|$. From this estimate, we deduce (43), using the definition of $\bar{A}$ and Fatou's lemma. The result follows. □

The operator $\bar{A}$ has the following monotonicity-like property :

Proposition 4. *Let $(u, v) \in \bar{A}$ and $(\hat{u}, \hat{v}) \in \bar{A}$. Then,*

$$(44) \qquad (v - \hat{v}, u - \hat{u})_{V,\Gamma} \ge \int_\Omega (f(u) - f(\hat{u}))(u - \hat{u})dx \ (\ge 0).$$

Moreover, for all $\lambda \ge 0$,

$$(45) \qquad \|u - \hat{u}\|_{H,\Gamma} \le \|L\,(u - \hat{u} + \lambda(v - \hat{v}))\|_{H',\Gamma} .$$

Proof. If $u \in D(A)$ and $\hat{u} \in D(A)$, then from (42), we deduce that

$$(Au - A\hat{u}, u - \hat{u})_{V,\Gamma} = \int_\Omega (f(u) - f(\hat{u}))(u - \hat{u})dx.$$

Estimate (44) follows from the definition of $\bar{A}$ and from Fatou's lemma. Let $(u, v) \in \bar{A}$, $(\hat{u}, \hat{v}) \in \bar{A}$ and $\lambda \ge 0$. By (41),

$$\begin{aligned} \|u - \hat{u}\|^2_{H,\Gamma} &= (u - \hat{u}, u - \hat{u})_{V,\Gamma} \\ &\le (u - \hat{u} + \lambda(v - \hat{v}), u - \hat{u})_{V,\Gamma} \end{aligned}$$

where, in the second line, we used (44). Thus, by (39),

$$\begin{aligned} \|u - \hat{u}\|^2_{H,\Gamma} &\le \langle L\,(u - \hat{u} + \lambda(v - \hat{v}))\,, u - \hat{u}\rangle_{H',H} \\ &\le \|L\,(u - \hat{u} + \lambda(v - \hat{v}))\|_{H',\Gamma} \|u - \hat{u}\|_{H,\Gamma} , \end{aligned}$$

and (45) is proved on dividing by $\|u - \hat{u}\|_{H,\Gamma}$. □

We next prove a maximality-like property of A. We first have

Lemma 2. *For all $\lambda > 0$ and for all $v \in H^2$, there exists a unique $u \in D(A) \cap H^2$ such that $u + \lambda Au = v$.*

Proof. Uniqueness is a consequence of (45). Existence is a consequence of a standard result on nonlinear elliptic equations. More precisely, let $\lambda > 0$. Then, for all $(u, v) \in D(A) \times H$,

$$u + \lambda Au = v \iff Lu + \lambda f(u) = Lv.$$

If v belongs to H^2, we can set $g = Lv \in L^2$ and the problem is to find $u \in D(A) \cap H^2$ such that

$$(46) \qquad u + V \cdot \nabla u - \mathrm{div}(\Gamma \nabla u) + \lambda f(u) = g.$$

A proof of this preliminary result is given in the Appendix. □

¿From the previous results, we deduce:

Theorem 4. *Let $\lambda > 0$. For all $v \in H$, there exists a unique $u \in D(\bar{A})$ such that $u + \lambda \bar{A} u \ni v$. Moreover, for all $(u, v) \in D(\bar{A}) \times H$ and for all $(\hat{u}, \hat{v}) \in D(\bar{A})$ such that $u + \lambda \bar{A} u \ni v$ and $\hat{u} + \lambda \bar{A} \hat{u} \ni \hat{v}$, we have*

$$\|u - \hat{u}\|_{H,\Gamma} \leq \|L(v - \hat{v})\|_{H',\Gamma}. \tag{47}$$

Proof. Estimate (47) is a direct consequence of estimate (45). It implies uniqueness. Let now $v \in H$ be fixed. We choose a sequence (v_n) in H^2 such that $v_n \to v$ in H. For every v_n, Lemma 2 provides $u_n \in D(A) \cap H^2$ which solves

$$u_n + \lambda A u_n = v_n. \tag{48}$$

By (47), the sequence $(u_n)_n$ is a Cauchy sequence in H. Therefore, it has a limit u in H. Letting n tend to $+\infty$ in (48) yields $u + \lambda \bar{A} u \ni v$. This proves existence, and it concludes the proof. □

When $V = 0$, relation (44) and Theorem 4 show that A is a maximal monotone operator in H for the scalar product $(\cdot, \cdot)_{0,\Gamma}$. In this case, A is in fact the subdifferential of the lower semi-continuous (l.s.c.) function Φ (see (13)). The proof is similar to the proof of Theorem 1.

Proposition 5. *If $V = 0$, then $A = \partial \Phi$ for the scalar product $(\cdot, \cdot)_{0,\Gamma}$ on H.*

3.2. **Global existence for the evolution problem.** In order to study the time-dependent problem, we first introduce a time dependent version of $\bar{A}$. We work in the vector space $\mathcal{V} := L^2_{loc}([0, +\infty); H)$ and we define the following nonlinear operator $\mathcal{A}$ in $\mathcal{V}$: for all $(u, v) \in \mathcal{V} \times \mathcal{V}$,

$$(u, v) \in \mathcal{A} \iff (u(t), v(t)) \in \bar{A} \text{ for a.e. } t \geq 0. \tag{49}$$

By (44), for all $(u, v) \in \mathcal{A}$, for all $(\hat{u}, \hat{v}) \in \mathcal{A}$ and for all $T > 0$, we have

$$\int_0^T (v(t) - \hat{v}(t), u(t) - \hat{u}(t))_{V,\Gamma} dt \geq 0. \tag{50}$$

Conversely, we have

Lemma 3. *If $(u, v) \in \mathcal{V} \times \mathcal{V}$ satisfy*

$$\int_0^T (v(t) - \hat{v}(t), u(t) - \hat{u}(t))_{V,\Gamma} dt \geq 0$$

for all $T > 0$ and for all $(\hat{u}, \hat{v}) \in \mathcal{A}$, then $(u, v) \in \mathcal{A}$.

Proof. Let $\lambda > 0$ and let $(\hat{u}, \hat{v}) \in \mathcal{A}$. Using (41) and the assumption, we see that for all $T > 0$,

$$\begin{aligned} \int_0^T \|u - \hat{u}\|^2_{H,\Gamma} dt &\leq \int_0^T (u - \hat{u} + \lambda(v - \hat{v}), u - \hat{u})_{V,\Gamma} dt \\ &\leq \int_0^T \langle L(u - \hat{u} + \lambda(v - \hat{v})), u - \hat{u} \rangle_{H',H} dt. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\left(\int_0^T \|u - \hat{u}\|^2_{H,\Gamma} dt \right)^{1/2} \leq \left(\int_0^T \|L(u - \hat{u} + \lambda(v - \hat{v}))\|_{H',\Gamma} \right)^{1/2}. \tag{51}$$

Now, we set $\lambda = 1$ and we choose $\hat{u} : [0, +\infty) \to D(\bar{A})$ such that $u(t) + v(t) \in \hat{u}(t) + \bar{A}\hat{u}(t)$ for a.e. $t \geq 0$. This is possible, by Theorem 4. Moreover, $\hat{u}$ belongs to $\mathcal{V}$, by estimate (47). For $\hat{v} \in \mathcal{V}$ defined by $u + v = \hat{u} + \hat{v}$, we have (by definition) $\hat{v}(t) \in \bar{A}\hat{u}(t)$ for a.e. $t \geq 0$. In particular, $(\hat{u}, \hat{v}) \in \mathcal{A}$. We may therefore apply (51), and we find $u = \hat{u}$ for a.e. $t \geq 0$. Thus, $v = \hat{v}$ for a.e. $t \geq 0$ and the proof is complete. □

Recall that $f : (a, b) \to \mathbb{R}$ is maximal monotone. Thus, for every $\lambda > 0$, the resolvent $j_\lambda = (I + \lambda f)^{-1}$ is well defined from $\mathbb{R}$ onto (a, b) [2]; j_λ is globally Lipschitz continuous on $\mathbb{R}$ (with Lipschitz constant equal to 1), and since f is continuous on (a, b), j_λ converges uniformly to the identity I on every compact subset of (a, b). We use the Yosida approximation of f defined for $\lambda > 0$ by

$$f_\lambda = (I - j_\lambda)/\lambda. \tag{52}$$

The function $f_\lambda : \mathbb{R} \to \mathbb{R}$ is nonincreasing and globally Lipschitz continuous. Moreover, $f_\lambda(r) = f(j_\lambda(r))$ for every $r \in \mathbb{R}$, and f_λ converges uniformly to f on every compact subset of (a, b).

Let $\lambda > 0$. We consider now the following regularization of problem (38):

$$u_t^\lambda + V \cdot \nabla u_t^\lambda - \operatorname{div}(\Gamma \nabla u_t^\lambda) + f_\lambda(u^\lambda) - \theta u^\lambda - \alpha \Delta u^\lambda = \gamma \quad x \in \Omega,\ t > 0, \tag{53}$$

and where $u^\lambda : \Omega \times [0, +\infty) \to \mathbb{R}$ satisfies the initial condition $u^\lambda(0, \cdot) = u_0^\lambda$. Applying J to the equation above, we write (53) in the following formal form: find $u^\lambda : [0, +\infty) \to H$ such that

$$u_t^\lambda + J f_\lambda(u^\lambda) - \theta J u^\lambda - \alpha J \Delta u^\lambda = J\gamma, \quad t > 0,\ u^\lambda(0) = u_0^\lambda. \tag{54}$$

In the remainder of this section, we assume, unless otherwise specified, that $\gamma \in L^2_{loc}([0, +\infty); H')$. We have

Proposition 6. *For every $u_0^\lambda \in H$, there exists a unique $u^\lambda \in W^{1,2}_{loc}([0, +\infty); H)$ which solves (54). If, moreover, $u_0^\lambda \in H^2$ and $\gamma \in L^2_{loc}([0, +\infty); L^2)$, then $u^\lambda \in W^{1,2}_{loc}([0, +\infty); H^2)$.*

Proof. The function f_λ is globally Lipschitz continuous from L^2 into L^2. As a consequence, the map $v \mapsto J f_\lambda(v) - \theta J v - \alpha J \Delta v$ is globally Lipschitz from H into H, and also from H^2 into H^2. The result follows from the standard Cauchy-Lipschitz theorem on the Banach space H or H^2. □

In order to let $\lambda \to 0$, we now derive a priori estimates which are independent of λ. For this purpose, we choose $c \in (a, b)$ as in the definition (12) of F and for every $\lambda > 0$, we define $F_\lambda(r) = \int_c^r f_\lambda(s) ds$; $F_\lambda : \mathbb{R} \to \mathbb{R}$ is a C^1 convex function.

Let $\varepsilon > 0$ be a small parameter to be determined below, and define

$$G_\lambda(v) := \frac{1}{2} \|v\|^2_{H,\Gamma} + \varepsilon \int_\Omega F_\lambda(v) + D_\lambda - \frac{\theta}{2} |v|^2 + \frac{\alpha}{2} |\nabla v|^2 dx,$$

where $D_\lambda = f_\lambda(c)c + 1$. We have

Lemma 4. *There exists $\lambda^+ > 0$ such that for all $\lambda \in (0, \lambda^+]$ and for all $v \in H$, $G_\lambda(v) \geq \|v\|^2_{H,\Gamma}/8$. Moreover, there exist two positive constant C_1 and C_2 independent of λ such that for all $\lambda \in (0, \lambda^+]$ and for all $t \geq 0$,*

$$\begin{aligned} G_\lambda(u^\lambda(t)) + \frac{\varepsilon}{4} \int_0^t \left\| u_t^\lambda(s) \right\|^2_{H,\Gamma} ds + \int_0^t \int_\Omega (f_\lambda(u^\lambda) - f_\lambda(c))(u^\lambda - c) dx\, ds \\ \leq e^{C_1 t} G_\lambda(u_0^\lambda) + \frac{C_2}{C_1}(e^{C_1 t} - 1) + e^{C_1 t} \int_0^t \|\gamma(s)\|^2_{H',\Gamma} ds. \end{aligned} \tag{55}$$

Proof. From (53), we have

$$L u_t^\lambda + f_\lambda(u^\lambda) - \theta u^\lambda - \alpha \Delta u^\lambda = \gamma, \quad \text{for a.e. } t > 0. \tag{56}$$

We subtract $f_\lambda(c)$ from (56), and multiply by $u^\lambda - c$ to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| u^\lambda \right\|^2_{H,\Gamma} + \int_\Omega (f_\lambda(u^\lambda) - f_\lambda(c))(u^\lambda - c) + \alpha |\nabla u^\lambda|^2 dx \\ &= \int_\Omega -(V \cdot \nabla u_t^\lambda) u^\lambda + c u_t^\lambda + \theta |u^\lambda|^2 - \theta c u^\lambda - f_\lambda(c)(u^\lambda - c) dx + \langle \gamma, u^\lambda - c \rangle_{H',H}. \end{aligned}$$

Using that $rs \le \varepsilon_1 r^2 + s^2/(4\varepsilon_1)$ for all $r, s \ge 0$ and for all $\varepsilon_1 > 0$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left\|u^\lambda\right\|_{H,\Gamma}^2 + \int_\Omega (f_\lambda(u^\lambda) - f_\lambda(c))(u^\lambda - c) + \alpha|\nabla u^\lambda|^2 dx \tag{57}$$
$$\le \varepsilon_1 |\nabla u_t^\lambda|_0^2 + \varepsilon_1 |u_t^\lambda|_0^2 + C_1\left\|u^\lambda\right\|_{H,\Gamma}^2 + C_2 + \frac{1}{2}\|\gamma\|_{H',\Gamma}^2 + |f_\lambda(c)|^2,$$

where C_1 is a constant which depends only on $|V|$, ε_1, θ, $|\Omega|$, and C_2 is a constant which depends only on $|c|$, $|\Omega|$, ε_1, and θ. On multiplying equation (56) by u_t^λ, and integrating on Ω, we find

$$\frac{1}{2}\left\|u_t^\lambda\right\|_{H,\Gamma}^2 + \frac{d}{dt}\left(\int_\Omega F_\lambda(u^\lambda) + D_\lambda - \frac{\theta}{2}|u^\lambda|^2 + \frac{\alpha}{2}|\nabla u^\lambda|^2 dx\right) \le \frac{1}{2}\|\gamma\|_{H',\Gamma}^2. \tag{58}$$

We now choose $\varepsilon \in (0,1)$ small enough so that $\varepsilon\theta \le 1/2$ and $\varepsilon f(c)^2 \le 1/8$. Since $f_\lambda(c) \to f(c)$ as $\lambda \to 0^+$, we have $\varepsilon f_\lambda(c)^2 \le 1/4$ for all $\lambda \in (0, \lambda^+]$, for some positive real number λ^+. By convexity, $F_\lambda(r) \ge f_\lambda(c)(r-c)$ for all $r \in \mathbb{R}$ (recall that $F_\lambda(c) = 0$). Thus,

$$r^2/8 + \varepsilon(F_\lambda(r) + D_\lambda) \ge r^2/8 + \varepsilon(1 + f_\lambda(c)r) \ge \varepsilon/2,$$

for all $r \in \mathbb{R}$. As a consequence,

$$G_\lambda(v) = \int_\Omega \frac{(1-\varepsilon\theta)}{2}|v|^2 + \varepsilon(F_\lambda(v) + D_\lambda) + \frac{1}{2}(\Gamma\nabla v)\cdot\nabla v + \frac{\varepsilon\alpha}{2}|\nabla v|^2 dx$$
$$\ge \frac{1}{8}\|v\|_{H,\Gamma}^2, \tag{59}$$

for all $v \in H$, as claimed. In estimate (57), we choose now $\varepsilon_1 > 0$ small enough so that

$$\varepsilon_1(|v|_0^2 + |\nabla v|_0^2) \le \frac{\varepsilon}{4}\|v\|_{H,\Gamma}^2 \quad \forall v \in H,$$

and we compute (57)+$\varepsilon\times$ (58). It yields

$$\frac{d}{dt}G_\lambda(u^\lambda) + \frac{\varepsilon}{4}\left\|u_t^\lambda\right\|_{H,\Gamma}^2 + \int_\Omega (f_\lambda(u^\lambda) - f_\lambda(c))(u^\lambda - c) + \alpha|\nabla u^\lambda|^2 dx$$
$$\le C_1\left\|u^\lambda\right\|_{H,\Gamma}^2 + C_2 + (1/2 + \varepsilon/2)\|\gamma\|_{H',\Gamma}^2 + |f_\lambda(c)|^2.$$

For $\lambda \in (0, \lambda^+]$, we have $|f_\lambda(c)| \le 1/(4\varepsilon)$, so that

$$\frac{d}{dt}G_\lambda(u^\lambda) + \frac{\varepsilon}{4}\left\|u_t^\lambda\right\|_{H,\Gamma}^2 + \int_\Omega (f_\lambda(u^\lambda) - f_\lambda(c))(u^\lambda - c)dx$$
$$\le C_1' G_\lambda(u^\lambda) + C_2' + \|\gamma\|_{H',\Gamma}^2, \tag{60}$$

where $C_1' = 2C_1$ and $C_2' = C_2 + 1/(4\varepsilon)$ are independent of λ. An application of Gronwall's lemma concludes the proof. □

We also have H^2 a priori estimates :

Lemma 5. *Assume that $\gamma \in L^2_{loc}([0,+\infty); L^2)$ and let $T > 0$. There exists a constant $C(T)$ independent of $\lambda \in (0, \lambda^+]$ such that for all $u_0^\lambda \in H^2$, the solution u^λ of* (54) *satisfies*

$$\left\|u^\lambda(t)\right\|_{H^2}^2 \le C(T)\left(G_\lambda(u_0^\lambda) + \left\|u_0^\lambda\right\|_{H^2}^2 + 1 + \int_0^T |\gamma(t)|_0^2 dt\right), \quad \forall t \in (0,T].$$

Proof. For a.e. $t > 0$, we take the L^2-scalar product of equation (56) by $-\Delta u^\lambda$. It reads

$$\int_\Omega -(Lu_t^\lambda)\Delta u^\lambda - f_\lambda(u^\lambda)\Delta u^\lambda + \alpha|\Delta u^\lambda|^2 dx = \int_\Omega \theta|\nabla u^\lambda|^2 - \gamma\Delta u^\lambda\, dx,$$

for a.e. $t > 0$. Using that

$$-\int_\Omega f_\lambda(u^\lambda)\Delta u^\lambda dx = \int_\Omega f'_\lambda(u^\lambda)|\nabla u^\lambda|^2 dx \geq 0,$$

and that

$$\int_\Omega \operatorname{div}(\Gamma\nabla v)\Delta w\, dx = \int_\Omega \operatorname{tr}\left(\nabla(\Gamma^{1/2}\nabla v))^t(\nabla(\Gamma^{1/2}\nabla w))\right) dx \quad \forall v, w \in H^2,$$

together with the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\nabla u^\lambda|_0^2 + |u^\lambda|_{H^2,\Gamma}^2\right) + \alpha|\Delta u^\lambda|_0^2$$

$$\leq |\Delta u^\lambda|_0^2 + \theta|\nabla u^\lambda|_0^2 + \frac{|V|^2}{2}|\nabla u_t^\lambda|^2 + \frac{1}{2}|\gamma|_0^2, \tag{61}$$

for a.e. $t > 0$, where

$$|v|_{H^2,\Gamma}^2 := \int_\Omega \operatorname{tr}\left(\nabla(\Gamma^{1/2}\nabla v))^t(\nabla(\Gamma^{1/2}\nabla v))\right) dx \quad \forall v \in H^2.$$

Let $T > 0$. Now, we compute (60) + $2\varepsilon_2\times$ (61) and for $\varepsilon_2 > 0$ small enough, we obtain

$$\frac{d}{dt}\left(G_\lambda(u^\lambda) + \varepsilon_2|\nabla u^\lambda|_0^2 + \varepsilon_2|u^\lambda|_{H^2,\Gamma}^2\right)$$
$$\leq C'_1 G_\lambda(u^\lambda) + 2\varepsilon_2|\Delta u^\lambda|_0^2 + 2\varepsilon_2\theta|\nabla u^\lambda|_0^2 + C'_2 + \varepsilon_2|\gamma|_0^2 + \|\gamma\|_{H',\Gamma}^2.$$

The norm $v \mapsto (\|v\|_{H,\Gamma}^2 + |v|_{H^2,\Gamma}^2)^{1/2}$ is equivalent to the standard H^2-norm on H^2 [13], so that using (59), we find

$$\frac{d}{dt}H^\lambda(u^\lambda) \leq C_3 H^\lambda(u^\lambda) + C_4(1 + |\gamma|_0^2),$$

with

$$H_\lambda(v) := G_\lambda(v) + \varepsilon_2|\nabla v|_0^2 + \varepsilon_2|v|_{H^2,\Gamma}^2 \geq C_5\|v\|_{H^2}^2 \quad \forall v \in H^2,$$

and where C_3, C_4 and C_5 are independent of λ. An application of Gronwall's lemma concludes the proof. □

We can now state:

Theorem 5. *Let $u_0 \in D(\Phi)$. Then there exists $u \in W_{loc}^{1,2}([0,+\infty); H)$ such that $u(t) \in D(\bar{A})$ for a.e. $t > 0$ and*

$$u_t + \bar{A}u - \theta Ju - \alpha J\Delta u \ni J\gamma, \quad \text{for a.e. } t > 0,\ u(0) = u_0. \tag{62}$$

If moreover, $u_0 \in H^2$ and $\gamma \in L_{loc}^2\left((0,\infty); L^2(\Omega)\right)$, then $u \in L^\infty(0,T;H^2)$ for all $T > 0$.

Proof. We set $u_0^\lambda = u_0$ and we consider the solution u^λ of problem (54) given by Proposition 6. Since $\Phi(u_0) < +\infty$, and since $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ is a l.s.c. convex fucntion, we have $\Phi_\lambda(u_0) := \int_\Omega F_\lambda(u_0) \to \Phi(u_0)$ as $\lambda \to 0^+$ (see [2][Proposition 2.11, page 39]). Thus, by Lemma 4, $(u^\lambda)_{\lambda\in(0,\lambda^+]}$ is bounded in $W^{1,2}(0,T;H)$ for all $T > 0$. There exists $u \in W_{loc}^{1,2}([0,+\infty); H)$ such that, up to a subsequence, $u^\lambda \to u$ weakly in $W^{1,2}(0,T;H)$ for all $T > 0$.

Now, recall that $f_\lambda(r) = f(j_\lambda(r))$ for all $r \in \mathbb{R}$, so that equation (54) can be written

$$-u_t^\lambda + \theta Ju^\lambda + \alpha J\Delta u^\lambda + J\gamma = Jf(j_\lambda(u^\lambda)), \quad \text{for a.e. } t > 0. \tag{63}$$

Let $(\hat{u}, \hat{v}) \in \mathcal{A}$ and $T > 0$. By (50), we have

$$I_\lambda := \int_0^T (-u_t^\lambda + \theta Ju^\lambda + \alpha J\Delta u^\lambda + J\gamma - \hat{v}, j_\lambda(u^\lambda) - \hat{u})_{V,\Gamma} dt \geq 0.$$

Writing $j_\lambda(u^\lambda) = u^\lambda - (u^\lambda - j_\lambda(u^\lambda))$ and using (63) again, we have $I_\lambda = I^1_\lambda - I^2_\lambda$, where

$$I^1_\lambda = \int_0^T (-u^\lambda_t + \theta J u^\lambda + \alpha J \Delta u^\lambda + J\gamma - \hat{v}, u^\lambda - \hat{u})_{V,\Gamma} dt$$

and

$$I^2_\lambda = \int_0^T (Jf(j_\lambda(u^\lambda)) - \hat{v}, u^\lambda - j_\lambda(u^\lambda))_{V,\Gamma} dt.$$

Recall that

$$u^\lambda - j_\lambda(u^\lambda) = \lambda f(j_\lambda(u^\lambda)) = \lambda f_\lambda(u^\lambda), \tag{64}$$

so that

$$I^2_\lambda = \lambda \int_0^T |f_\lambda(u^\lambda)|^2 dt - \int_0^T (\hat{v}, u^\lambda - j_\lambda(u^\lambda))_{V,\Gamma} dt.$$

By (63), $Jf_\lambda(u^\lambda)$ is bounded in $L^2(0,T;H)$. From (64), we deduce that $u^\lambda - j_\lambda(u^\lambda)$ converges to 0 in $L^2(0,T;H')$. On the other hand, $(u^\lambda)_\lambda$ is bounded in $L^2(0,T;H)$ and since j_λ is a 1-Lipschitz continuous function with $(j_\lambda(c))_\lambda$ bounded, $(j_\lambda(u^\lambda))_\lambda$ is also bounded in $L^2(0,T;H)$. Thus, $u^\lambda - j_\lambda(u^\lambda)$ converges weakly to 0 in $L^2(0,T;H)$. As a consequence, $\liminf I^2_\lambda \geq 0$.

Next, we write $I^1_\lambda = K^1_\lambda + K^2_\lambda + K^3_\lambda$ with

$$\begin{aligned} K^1_\lambda &= \int_0^T (-u^\lambda_t + \theta J u^\lambda + \alpha J\Delta u^\lambda, u^\lambda)_{V,\Gamma} dt, \\ K^2_\lambda &= -\int_0^T (-u^\lambda_t + \theta J u^\lambda + \alpha J\Delta u^\lambda, \hat{u})_{V,\Gamma} dt, \\ K^3_\lambda &= \int_0^T (J\gamma - \hat{v}, u^\lambda - \hat{u})_{V,\Gamma} dt. \end{aligned}$$

Clearly, K^2_λ tends to $-\int_0^T (-u_t + \theta J u + \alpha J\Delta u, \hat{u})_{V,\Gamma} dt$ as λ tends to 0^+, and K^3_λ tends to $\int_0^T (J\gamma - \hat{v}, u - \hat{u})_{V,\Gamma} dt$. On the other hand,

$$K^1_\lambda = -\int_0^T \frac{1}{2}\frac{d}{dt} \left\|u^\lambda\right\|^2_{H,\Gamma} dt - \int_0^T (V \cdot \nabla u^\lambda_t) u^\lambda dt + \int_0^T (\theta J u^\lambda, u^\lambda)_{V,\Gamma} dt$$
$$-\alpha \int_0^T |\nabla u^\lambda|^2 dt.$$

The space $W^{1,2}(0,T;H)$ is compactly imbedded into $L^2(0,T;L^2)$ [12], so that u^λ converges strongly to u in $L^2(0,T;L^2)$. Thus, the second and the third integral above have the limit we expect as λ tends to 0. Moreover,

$$\begin{aligned} \limsup \left(-\int_0^T \frac{d}{dt} \left\|u^\lambda\right\|^2_{0,\Gamma} dt \right) &= \|u_0\|^2_{H,\Gamma} - \liminf \left\|u^\lambda(T)\right\|^2_{H,\Gamma} \\ &\leq \|u_0\|^2_{H,\Gamma} - \|u(T)\|^2_{H,\Gamma} \\ &= -\int_0^T \frac{d}{dt} \|u\|^2_{0,\Gamma} dt = 2\int_0^T (-u_t, u)_{V,\Gamma} dt. \end{aligned}$$

In the last inequality, we used that the space $W^{1,2}(0,T;H)$ is compactly imbedded into $C^0([0,T],L^2)$ and that $\sup_{t\in[0,T]} \left\|u^\lambda(t)\right\|_{H,\Gamma}$ is bounded by a constant independent of λ, so that $u^\lambda(t)$ tends weakly to $u(t)$ in H, for every $t \in [0,T]$. We also have

$$\limsup \left(-\alpha \int_0^T |\nabla u^\lambda|^2 dt \right) = -\alpha \liminf \left(\int_0^T |\nabla u^\lambda|^2 dt \right) \leq -\alpha \int_0^T |\nabla u|^2 dt.$$

Summing up, we have proved that

$$I \geq \limsup I^1_\lambda \geq \limsup I^2_\lambda \geq 0,$$

where

$$I = \int_0^T (-u_t + \theta Ju + \alpha J\Delta u + J\gamma - \hat{v}, u - \hat{u})_{V,\Gamma} dt.$$

This is true for all $T > 0$ and for all $(\hat{u}, \hat{v}) \in \mathcal{A}$ so we can apply Lemma 3, and we find that $u(t) \in D(\bar{A})$ for a.e. $t > 0$, and that u solves (62) as claimed.

If, moreover, $u_0 \in H^2$, then for all $T > 0$, $(u^\lambda)_\lambda$ is bounded in $C^0([0,T]; H^2)$ by Lemma 5. In this case, u belongs to $L^\infty(0,T;H^2)$ for all $T > 0$. The proof is complete. □

When $V = 0$, by application of maximal monotone operator theory, we have

Proposition 7. *If $V = 0$ and $u_0 \in D(\Phi)$, there exists a unique solution u of* (62) *in the sense of Theorem 5.*

Proof. The operator $\bar{A}$ is maximal monotone in H (Proposition 5) and the linear operator $v \mapsto -\theta Jv - \alpha J\Delta u$, which is bounded in H, is a Lipschitz perturbation of $\bar{A}$. The result follows [2]. □

3.3. **Asymptotic behaviour ($V = 0$, $\gamma = 0$, $\alpha > 0$).** In this section, we consider equation (62) in the autonomous case $\gamma = 0$. We also assume $V = 0$ (in order to have a uniqueness principle) and (unless otherwise specified) $\alpha > 0$ (this corresponds to a dissipativity assumption). The parameter θ is nonnegative.

With these assumptions, $\bar{A} = \partial\Phi$ for the scalar product $(\cdot,\cdot)_{0,\Gamma}$ on H and the solution $u \in W^{1,2}_{loc}([0,+\infty); H)$ given by Theorem 5 satisfies

$$u \in D(\bar{A}) \text{ and } u_t + \bar{A}u - \theta Ju - \alpha J\Delta u = 0 \quad \text{for a.e. } t > 0,\ u(0) = u_0. \tag{65}$$

The theory of maximal monotone operators garantees that for every $u_0 \in \overline{D(\bar{A})}$, equation (65) has a unique strong solution $u \in C([0,+\infty); H)$ such that $u(t) \in D(\bar{A})$ for all $t > 0$ [2]. Equation (65) defines a dynamical system $\{S_t\}_{t\geq 0}$ on $\overline{D(\bar{A})}$ through $S_t u_0 = u(t)$ for all $t \geq 0$.

We will prove that under appropriate assumptions, a solution converges to a stationary solution. By definition, $u_0 \in \overline{D(\bar{A})}$ is a stationary solution for $\{S_t\}$ if $S_t u_0 = u_0$ for all $t \geq 0$. By regularization, any such stationary solution u_0 belongs to $D(\bar{A})$; by derivation $(S_t u_0)_t = 0$ for a.e. $t > 0$. Thus,

$$\mathcal{S} = \{v \in D(\bar{A}) \ : \ 0 \in \bar{A}v - \theta Jv - \alpha J\Delta v\}$$

is the set of stationary solutions for equation (65).

As a generalization of Proposition 1, we have

Proposition 8. *Assume that $\alpha \geq 0$ and $\theta \geq 0$. If the function $s \mapsto f(s) - \theta s$ has no root in (a,b), then $\mathcal{S} = \emptyset$. Otherwise, every $u_0 \in \mathcal{S}$ satisfies $u_0 \in [\underline{c}, \overline{c}]$ a.e. in Ω, where*

$$\underline{c} = \inf\{c \in (a,b) \ : \ f(c) \geq \theta c\} \quad \textit{and} \quad \overline{c} = \sup\{c \in (a,b) \ : \ f(c) \leq \theta c\}.$$

Proof. First assume that $s \mapsto f(s) - \theta s$ has a root in (a,b) (then $\underline{c} \leq \overline{c}$), and let $u_\star \in \mathcal{S}$. We first show that $u \leq \overline{c}$ a.e. in Ω. If $\overline{c} = +\infty$, this is obvious, so we can assume $\overline{c} < +\infty$. By definition, there exists a sequence $u_n \in D(A)$ such that $u_n \to u_\star$ in H and a.e. in Ω and

$$T_n := f(u_n) - \theta u_n - \alpha\Delta u_n \to 0 \text{ in } H'.$$

Let $(s)_+ := \max\{0, s\}$. The map $w_n := (u_n - \overline{c})_+$ belongs to H and $w_n \to (u_\star - \overline{c})_+$ in H and a.e. in Ω. The sequence $(f(u_n) - \theta u_n)w_n \geq 0$ converges a.e. in Ω to $(f(u_\star) - \theta u_\star)(u_\star - \overline{c})_+ \geq 0$. By Fatou's lemma,

$$\int_\Omega (f(u_\star) - \theta u_\star)(u_\star - \overline{c})_+ + \alpha|\nabla(u_\star - \overline{c})_+|^2 dx \leq \lim_{n\to+\infty} \langle T_n, w_n\rangle_{H',H} = 0.$$

Thus, $u_\star \leq \overline{c}$ a.e. in Ω as claimed. We prove similarly that $u_\star \geq \underline{c}$ a.e. in Ω.

If $f(s) - \theta s > 0$ for all $s \in (a, b)$, then we replace $\overline{c}$ by any $c \in (a, b)$ in the argument above, and we find that $u_\star \leq c$ a.e. in Ω, for all $c \in (a, b)$. This is not possible since $u_\star \in (a, b)$ a.e. in Ω, so $\mathcal{S}$ is empty. The case where $f(s) - \theta s < 0$ for all $s \in (a, b)$ is treated similarly. □

In the remainder of this section, we assume that $s \mapsto f(s) - \theta s$ has a root in (a, b); without loss of generality, we can assume that

$$0 \in (a, b) \text{ and } f(0) = 0. \tag{66}$$

We also assume that there exist two constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$f(s)s - \theta s^2 \geq c_1|s| - c_2 \quad \forall s \in (a, b). \tag{67}$$

Estimate (67) is a automatically satisfied if (a, b) is bounded. If $b = +\infty$ or $a = +\infty$, it is a coercivity condition which can be written in the following equivalent form:

$$(b = +\infty) \quad \Rightarrow \quad \left(\liminf_{s\to+\infty} (f(s) - \theta s) > 0\right),$$

$$(a = -\infty) \quad \Rightarrow \quad \left(\limsup_{s\to-\infty} (f(s) - \theta s) < 0\right).$$

Note that, under assumptions (66-67), $[\underline{c}, \overline{c}] \subset (a, b)$. By choosing c_1 smaller if necessary, we see that assumption (67) implies

$$F(s) - \theta s^2/2 \quad \geq \quad c_1|s| - c_3 \quad \forall s \in (a, b), \tag{68}$$

for some nonnegative constant c_3. As a shortcut, we define

$$E(v) := \Phi(v) - \frac{\theta}{2}|v|_0^2 + \alpha|\nabla v|_0^2, \quad \forall v \in H.$$

Lemma 6. *Assume that $\alpha > 0$, $\theta \geq 0$ and that f satisfies* (66)-(67). *Let $u_0 \in H$ such that $\Phi(u_0) < +\infty$ and let u be the unique solution in $W^{1,2}_{loc}([0, +\infty); H)$ of*

$$u(t) \in D(\bar{A}) \quad \text{and} \quad u_t + \bar{A}u - \theta Ju - \alpha J\Delta u \ni 0 \quad \text{for a.e. } t > 0, \quad u(0) = u_0. \tag{69}$$

Then u is bounded in H and

$$E(t) + \int_s^t \|u_t(\sigma)\|^2_{H,\Gamma}\, d\sigma \leq E(s) \quad \text{for all } 0 \leq s \leq t < +\infty. \tag{70}$$

Moreover, if $u_0 \in H^2$, then u is bounded in H^2.

Proof. We use the Yosida regularization of f (see (52)). Let $u^\lambda \in C^1([0, +\infty); H)$ solve (see Proposition 6)

$$Lu^\lambda_t + f_\lambda(u^\lambda) - \theta u^\lambda - \alpha\Delta u^\lambda \ni 0, \qquad \text{for a.e. } t > 0; \quad u^\lambda(0) = u_0. \tag{71}$$

First, we take the L^2-scalar product of (71) by u^λ_t and we integrate on $(0, t)$. We obtain

$$\int_0^t \left\|u^\lambda_t(s)\right\|^2_{H,\Gamma} ds + \int_\Omega F_\lambda(u^\lambda(t))dx - \frac{\theta}{2}|u^\lambda(t)|_0^2 + \frac{\alpha}{2}|\nabla u^\lambda(t)|_0^2$$
$$= \int_\Omega F_\lambda(u_0)dx - \frac{\theta}{2}|u_0|^2 + \frac{\alpha}{2}|\nabla u_0|_0^2.$$

Recall that, for all $T > 0$, the sequence (u^λ) converges (up to a subsequence) weakly in $W^{1,2}(0,T;H)$ to the solution u of (69) (see the proof of Theorem 5). For all $t \geq 0$, $u^\lambda(t)$ converges also (up to a subsequence) weakly in H, strongly in L^2 and a.e. in Ω to $u(t)$. Using Fatou's lemma and the lower semi-continuity of the H-norm, we can let $\lambda \to 0^+$ in the equation above and we obtain estimate (70) for $s = 0$ and $t \geq 0$. By uniqueness of the solution, estimate is therefore true for all $0 \leq s \leq t$. Moreover, by the coercivity condition (67),

$$c_1 \int_\Omega |u(t)| dx + \alpha |\nabla u(t)|_0^2 \leq E(u(t)) + c_3|\Omega| \leq E(u_0) + c_3|\Omega|, \quad \forall t \geq 0.$$

By the Poincaré inequality, $|v|_0 \leq C(\int_\Omega |v|dx + |\nabla v|_0)$ for all $v \in H$, for some constant $C > 0$. Thus, u is bounded in H. This proves the first assertion.

Assume now that $u_0 \in H^2$. We take the L^2-scalar product of equation (71) by u^λ and $-\Delta u^\lambda$ respectively, we add both results. We find

$$\frac{1}{2}\frac{d}{dt}E_1(u^\lambda) + \alpha|\nabla u^\lambda|_0^2 + \int_\Omega f_\lambda(u^\lambda)u^\lambda dx + \alpha|\Delta u^\lambda|_0^2 \leq \theta|u^\lambda|_0^2 + \theta|\nabla u^\lambda|_0^2,$$

where

$$E_1(v) = \|v\|_{H,\Gamma}^2 + |\nabla v|_0^2 + |v|_{H^2,\Gamma}^2 \quad (v \in H^2).$$

Arguing as previously, we can pass to the limit in the estimate above, and we obtain:

$$\frac{1}{2}\frac{d}{dt}E_1(u) + \alpha|\nabla u|_0^2 + \int_\Omega f(u)u\, dx + \alpha|\Delta u|_0^2 \leq \theta|u|_0^2 + \theta|\nabla u|_0^2 \tag{72}$$

in the sense of distributions in $(0, +\infty)$. Next, we use that u is bounded in H and that E_1 is bounded from above by the H^2 norm, and we find

$$\frac{d}{dt}E_1(u) + \varepsilon_3 E_1(u) \leq C,$$

for some constants $\varepsilon_3 > 0$ and $C > 0$. By the Gronwall lemma, $E_1(u(t)) \leq E_1(u_0) + C'$ for all $t \geq 0$. This concludes the proof. □

Remark 9. Similar estimates could be obtained in the case $V \neq 0$, with a slightly stronger coercivity assumption on f. However, by a lack of uniqueness principle, we are not able to prove that E is nonincreasing when $V \neq 0$.

Using Lemma 6 and the same arguments as in the proof of Theorem 3, we obtain

Theorem 6. *Assume that* $D(\Phi) = \overline{D(\bar{A})}$, *that* Φ *is continuous on* $D(\Phi)$ *and that* f *satisfies* (66)-(67). *For all* $u_0 \in D(\Phi) \cap H^2$, $\omega(u_0)$ *is a nonempty compact connected subset of* $\mathcal{S}$ *and* $d(S_t u_0, \omega(u_0)) \to 0$ *as* $t \to +\infty$.

As a consequence, by the same arguments as in the proof of Corollary 2, we have

Corollary 3. *Assume that the assumptions of Theorem 6 are satisfied. If* $N = 1$, 2 *or* 3 *and if* $u_0 \in D(\Phi) \cap H^2$, *then for* t *large enough,* $S_t u_0$ *takes values in a compact subset of* (a, b) *a.e. in* Ω *so that equation (38) is satisfied exactly.*

Remark 10. If $(a, b) = (-1, 1)$ and if f is the standard logarithmic nonlinearity given by (2), then the separation property of Corollary 3 applies. By using a Lojasiewicz-Simon inequality and arguing as in [1], we can prove that if $N = 1$, 2 or 3 and if $u_0 \in D(\Phi) \cap H^2$, then $S_t u_0$ converges to a stationary solution $u_\star \in \mathcal{S}$ in H^r for $1 \leq r < 2$. A similar result holds if $(a, b) = \mathbb{R}$ and F is a convex polynomial with at most critical growth (cf. (34)).

4. Appendix

We recall here the various tools and results based on the H^1-capacity and which are used in this paper. Proofs may be found in [3, 9, 11].

Definition of the capacity: Let us recall the definition of the H^1-capacity: for a compact subset $K \subset \mathbb{R}^N$

$$cap(K) = \inf\{\int_{\mathbb{R}^N} \zeta^2 + |\nabla\zeta|^2; \zeta \in C_0^\infty(\mathbb{R}^N),\ \zeta \geq 1 \text{ on } K\}.$$

For any $\omega \subset \mathbb{R}^N$ open, and for any $E \subset \mathbb{R}^N$

$$cap(\omega) = \sup\{cap(K);\ K \subset \omega,\ K \text{ compact}\},\ cap(E) = \inf\{cap(\omega);\ E \subset \omega\}.$$

Quasi-continuous representation: In dimension $N = 1$, the capacity of a point is positive and any element of $H^1_{loc}(\mathbb{R})$ has a unique continuous representation. The corresponding result in any dimension is that any $u \in H^1_{loc}(\mathbb{R}^N)$ has a *quasi-continuous representation* (we will still denote it u) which is unique up to a set of zero-capacity. And *quasi-continuous* means that, for all $\varepsilon > 0$, there exists an open subset ω_ε with $cap(\omega_\varepsilon) < \varepsilon$ such that the restriction of u to $\mathbb{R}^N \setminus \omega_\varepsilon$ is continuous. In particular, $cap([|u| = \infty]) = 0$. Moreover, if u_n converges to u in $H^1(\Omega)$, there exists a subsequence of u_n which converges to u *quasi-everywhere* (which means everywhere except on a set of zero capacity).

Lemma 7. Properties of H'-measures, see [3, 9].

- *If $\mu \in H' \cap \mathcal{M}$, then $|\mu|(E) = 0$ for all measurable $E \subset \mathbb{R}^N$ with $cap(E) = 0$.*
- *Let $m \in L^1(\Omega) \cap H', w \in H$ and assume $m\, w \geq 0$ a.e.. Then,*

$$m\, w \in L^1(\Omega), \text{ and } < m, w >_{H' \times H} = \int_\Omega m\, w.$$

- *More generally, if $\mu \in \mathcal{M} \cap H', w \in H, m\, w \geq -g\,|\mu|$ a.e. where $g, m \in L^\infty(\overline{\Omega}, d|\mu|)$ and $\mu = m|\mu|$, then*

$$w \in L^1(\overline{\Omega}, d|\mu|) \text{ and } < \mu, w >_{H' \times H} = \int_{\overline{\Omega}} w\, d\mu.$$

Proof of the existence in (17) *and in its generalization* (46)*:* We may assume $\lambda = 1$ in (46). Let $G \in L^2(\Omega)$. We need to solve

$$u \in D(A) \cap H^2,\ Lu + f(u) = G. \tag{73}$$

Let us first replace f by its Yosida approximation $f_\mu := (I - j_\mu)/\mu, j_\mu := (I + \mu f)^{-1},$, where $\mu > 0$, and let us prove the existence of a solution to

$$Lu_\mu + f_\mu(u_\mu) = G. \tag{74}$$

This equation may be rewritten

$$\mu L u_\mu + u_\mu = \mu G + j_\mu(u_\mu).$$

Let T be the mapping which to $v \in L^2(\Omega) \subset H'$ associates the solution of

$$u \in H,\ \mu L u + u = \mu G + j_\mu(v).$$

If $\hat{u} = T(\hat{v})$, multiplying the difference $\mu L(u - \hat{u}) + (u - \hat{u}) = j_\mu(v) - j_\mu(\hat{v})$ by $u - \hat{u}$ gives (recall that j_μ is a contraction from $\mathbb{R}$ into itself):

$$\begin{aligned} \int_\Omega (\mu+1)(u-\hat{u})^2 + \mu(\Gamma\nabla(u-\hat{u}))\cdot\nabla(u-\hat{u}) &= \int_\Omega (u-\hat{u})(j_\mu(v) - j_\mu(\hat{v})) \\ &\leq \|u-\hat{u}\|_{L^2}\|v-\hat{v}\|_{L^2}. \end{aligned}$$

Thus T is a strict contraction for the L^2-norm, whence existence and uniqueness of $u_\mu \in H$ satisfying the equation in (74). Moreover u_μ is bounded in H as $\mu \to 0$ since by a monotonicity-like property, for $c \in (a,b)$, $\|u_\mu - c\|_{H,\Gamma} \leq \|G - Lc - f_\mu(c)\|_{H',\Gamma}$ (cf. (40)) and

$$\|f_\mu(c)\|_{H',\Gamma} = \|f_\mu(c)\|_{L^2(\Omega)} \to \|f(c)\|_{L^2(\Omega)} < +\infty \text{ as } \mu \to 0.$$

The H^2-estimate is obtained by multiplying the equation (74) by $-\Delta u_\mu$ to obtain

$$\int_\Omega -Lu_\mu \Delta u_\mu + |\nabla u_\mu|^2 f'_\mu(u_\mu) = \int_\Omega G(-\Delta u_\mu).$$

We use (see the proof of Lemma 5)

$$\int_\Omega -Lu_\mu \Delta u_\mu = \int_\Omega |\nabla u_\mu|^2 - V.\nabla u_\mu \Delta u_\mu + tr\left(\nabla(\Gamma^{1/2}\nabla u)^t \nabla(\Gamma^{1/2}\nabla u)\right).$$

As already seen in the proof of Lemma 5, $u \to \|u\|_H^2 + \int_\Omega tr\left(\nabla(\Gamma^{1/2}\nabla u)^t \nabla(\Gamma^{1/2}\nabla u)\right)$ is equivalent to the square of the H^2-norm. Therefore, using a standard Young's inequality for the two integrals $\int_\Omega G\Delta u_\mu, \int_\Omega V \cdot \nabla u_\mu \Delta u_\mu$ in the two last identities leads to an estimate of u_μ in H^2 as $\mu \to 0$.

Up to a subsequence, we may assume that, as $\mu \to 0$, u_μ converges weakly in H^2, strongly in H to some $u \in H \cap H^2$, and that $f_\mu(u_\mu)$ converges weakly in L^2 to $w := G - Lu$. We remember that $f_\mu(u_\mu) = f(j_\mu(u_\mu))$ and $\|j_\mu(u_\mu) - u_\mu\|_{L^2} = \mu\|f_\mu(u_\mu)\|_{L^2} \to 0$. By maximal monotonicity of f (or more precisely of the extension of f to $L^2(\Omega)$), we deduce that $w = f(u)$. Hence, we may pass to the limit in (74) and obtain that (73) has a solution. □

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Michel Pierre
ENS Cachan Bretagne, IRMAR, EUB
Campus de Ker Lann
35170 Bruz (France)

michel.pierre@bretagne.ens-cachan.fr

Morgan Pierre
Université de Poitiers,
Laboratoire de Mathématiques et Applications UMR CNRS 7348,
Téléport 2 - BP 30179,
Boulevard Marie et Pierre Curie,
86962 Futuroscope Chasseneuil (France)
morgan.pierre@math.univ-poitiers.fr