

THE INSTANTANEOUS LIMIT FOR REACTION-DIFFUSION SYSTEMS WITH A FAST IRREVERSIBLE REACTION

DIETER BOTHE

Center of Smart Interfaces
Technical University Darmstadt
Petersenstr. 32, 64287 Darmstadt, GERMANY

MICHEL PIERRE

ENS Cachan Bretagne, IRMAR, UEB
Campus de Ker Lann, 35170 Bruz, FRANCE

(Communicated by the associate editor name)

ABSTRACT. We consider reaction-diffusion systems which, in addition to certain slow reactions, contain a fast irreversible reaction in which chemical components A and B form a product P . In this situation and under natural assumptions on the RD-system we prove the convergence of weak solutions, as the reaction speed of the irreversible reaction tends to infinity, to a weak solution of a limiting system. The limiting system is a Stefan-type problem with a moving interface at which the chemical reaction front is localized.

1. Introduction. If a fast irreversible reaction of type $A + B \rightarrow P$ occurs between mobile species A and B , the reaction mainly happens close to the interface at which A and B are both present with similar concentrations. In the limit of an infinite reaction speed, the chemical reaction only happens at the interface where A and B are in contact and coexistence of both reactants is impossible, i.e. $c_A c_B = 0$, where c_i denotes the molar concentration of species i . Hence, this leads to the occurrence of a chemical reaction interface, a free boundary problem in mathematical terms. To be more specific, if mobile species react according to $A + B \rightarrow P$ inside an isolated bounded domain Ω , the reaction-diffusions system reads as

$$\left\{ \begin{array}{l} \partial_t c_A - d_A \Delta c_A = -k c_A c_B \\ \partial_t c_B - d_B \Delta c_B = -k c_A c_B \end{array} \right\} \text{ on } (0, +\infty) \times \Omega, \quad (1)$$
$$\partial_n c_A|_{\partial\Omega} = \partial_n c_B|_{\partial\Omega} = 0, \quad c_A|_{t=0} = c_{A,0}, \quad c_B|_{t=0} = c_{B,0}$$

in the simplest case and for mass action kinetics with finite reaction rate. Integration of one of the molar mass balances in (1) shows that if a limit for $k \rightarrow \infty$ should exist then necessarily $c_A c_B = 0$ a.e. on Ω , hence a spatial segregation occurs. Taking the difference of the two differential equations in (1) and using the fact that solutions are nonnegative formally suggests the limit system

$$\partial_t v - \Delta \phi(v) = 0 \text{ on } (0, +\infty) \times \Omega, \quad \partial_n \phi(v)|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0. \quad (2)$$

2000 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35B25, 92E20.

Key words and phrases. Chemical reaction interface, instantaneous chemical reaction, spatial segregation, Stefan-type problem.

Here the new scalar v is related to A and B by means of $c_A = v^+ := \max\{0, v\}$ and $c_B = v^- := \max\{0, -v\}$. Furthermore, $v_0 = c_{A,0} - c_{B,0}$ and

$$\phi(r) = \begin{cases} d_A r & \text{if } r \geq 0 \\ d_B r & \text{if } r < 0. \end{cases} \quad (3)$$

Equation (2) is similar to the classical Stefan problem in its enthalpy formulation, but here we have zero latent heat since the function ϕ has no plateau. In any case, this nonlinear diffusion equation is a special case of the so-called filtration equation (also called generalized porous medium equation); cf. [24]. In the present situation we have a very special function ϕ with piecewise constant derivative, but the different domains are not a priori known. In fact, the interface $\{v = 0\}$ is free and unknown. While existence, uniqueness and stability of weak solutions to (2) has already been shown in [11], [12], information about the regularity of the interface has been provided much later in [23].

Now, having a well-posed limit system, a natural question is whether and in which sense solutions of (1) converge to a solution of (2). For the non-singular case $c_{A,0} c_{B,0} = 0$, convergence in L^2 of any sequence of weak solutions (c_A^k, c_B^k) for $k \rightarrow \infty$ has been shown in [16]. This was improved in [5] to the case of general bounded initial values. Notice that for initial values with $c_{A,0} c_{B,0} > 0$, a temporal boundary layer near $t = 0$ is formed for large k .

In [13], the same limit problem has been studied for slightly more complex systems of type

$$\left\{ \begin{array}{l} \partial_t c_A - d_A \Delta c_A = c_A f_A(c_A) - k c_A c_B \\ \partial_t c_B - d_B \Delta c_B = c_B f_B(c_B) - \alpha k c_A c_B \end{array} \right\} \text{ on } (0, +\infty) \times \Omega, \quad (4)$$

$$\partial_n c_A|_{\partial\Omega} = \partial_n c_B|_{\partial\Omega} = 0, \quad c_A|_{t=0} = c_{A,0}, \quad c_B|_{t=0} = c_{B,0}$$

with $\alpha > 0$ and functions $f_j \in C^1(\mathbb{R}_+)$ such that $f_j(s) > 0$ on $(0, 1)$ and $f_j(s) < 0$ for all $s > 1$ and $j = A, B$. Note that the latter sign condition on f_A, f_B implies, by maximum principle, that, for bounded initial data, the (nonnegative) solutions c_A, c_B are uniformly bounded on $(0, T) \times \Omega$ for all $T < +\infty$, independently of k . In other words, we may consider that (4) is a perturbation of (1) by bounded nonlinearities (this will not be the case for our systems (5) or (10) below). Note also that w.l.o.g. we may assume $\alpha = 1$, since $\alpha > 0$ can be absorbed into c_A or c_B . More information about variants of this class of competition-diffusion systems and further references can be found in [17].

In the present paper, we are interested in the instantaneous limit for reaction-diffusion systems which contain a fast irreversible reaction of type $A + B \rightarrow P$, but also additional slow processes like other chemical reactions or macroscopic convection terms, where the point is to allow for quadratic growth of these. More precisely, we consider the following system of reaction-diffusion equations as a prototype model.

$$\left\{ \begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = -k u_1 u_2 + f_1(t, u) \\ \partial_t u_2 - d_2 \Delta u_2 = -k u_1 u_2 + f_2(t, u) \\ \partial_t u_i - d_i \Delta u_i = f_i(t, u) \quad (i = 3, \dots, n) \end{array} \right\} \text{ on } (0, +\infty) \times \Omega, \quad (5)$$

$$\partial_n u_i|_{\partial\Omega} = 0, \quad u_i|_{t=0} = u_{i,0}.$$

Here $u = (u_1, \dots, u_n)$ where u_1 and u_2 denote the molar concentrations of A and B , respectively, and for $i \geq 3$ the quantity u_i is the concentration of further components

which are involved in the chemical reaction network. The d_i ($i = 1, \dots, n$) are positive diffusion coefficients, $k > 0$ is the rate constant of the irreversible reaction and $u_{i,0}$ are the initial concentrations which we assume to be nonnegative and at least integrable. Finally, Ω is a given bounded open subset of \mathbb{R}^N with sufficiently smooth boundary ($\partial\Omega \in C^{2+\epsilon}$, say) and ∂_n denotes the outer normal derivative to $\partial\Omega$. Note that possible stoichiometric factors $\alpha, \beta > 0$ in front of the fast reaction term can be absorbed into u_1, u_2 , therefore we chose them to be equal to 1 without loss of generality.

Throughout the paper, we always assume that the nonlinearity $f : [0, T] \times \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ is jointly continuous and locally Lipschitz continuous in the second variable. Here, we concentrate on the case where f has *at most quadratic growth*, i.e.

$$|f(t, u)| \leq K(\psi_0 + |u|^2) \quad \text{with some } K > 0, \psi_0 \in L^1(Q_T), \quad (6)$$

and we assume that f allows *L^2 -control of the total mass*, by which we mean

$$\begin{cases} \langle f(t, u), e \rangle \leq L(\phi_0 + \langle u, e \rangle) & \text{with some } L > 0, e \gg 0 \text{ and} \\ \phi_0 \in L^1(Q_T), \Phi_0 \in L^2(Q_T) & \text{where } \Phi_0(t, x) = \int_0^t \phi_0(s, x) ds. \end{cases} \quad (7)$$

In (7), the notation $e \gg 0$ is short for $e \in \mathbb{R}^n$ with $e_i > 0$ for all i .

Finally, since we are only interested in nonnegative concentrations, we assume that f is *quasi-positive*, which means

$$f_i(t, u) \geq 0 \text{ whenever } u \in \mathbb{R}_+^n \text{ satisfies } u_i = 0. \quad (8)$$

The assumptions (6), (7) and (8) are satisfied for large classes of concrete systems. Let us only mention that (6) allows for bimolecular reactions with mass action kinetics, which is the standard case for elementary chemical reactions, condition (7), which is interesting in applications even with $L = 0$, follows from conservation of atoms if the species represent real chemical substances (cf. [15]) and (8) holds under the natural assumption that any consumption of species i is stopped if i is no longer present.

Setting $v = u_1 - u_2$ as before and $w = (u_3, \dots, u_n)$, the limit system for $k \rightarrow \infty$ presumably reads as

$$\begin{cases} \partial_t v - \Delta \phi(v) = g(t, v, w) \\ \partial_t w - D \Delta w = h(t, v, w) \end{cases} \quad \text{on } (0, +\infty) \times \Omega, \quad (9)$$

$$\partial_n \phi(v)|_{\partial\Omega} = 0, \partial_n w|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0, w|_{t=0} = w_0.$$

Here ϕ is given as in (3), $D = \text{diag}(d_3, \dots, d_n)$ with $d_i > 0$,

$$g(t, v, w) = f_1(t, v^+, v^-, w) - f_2(t, v^+, v^-, w)$$

and

$$h(t, v, w) = (f_3(t, v^+, v^-, w), \dots, f_n(t, v^+, v^-, w)).$$

Before looking at the limit $k \rightarrow +\infty$, we first show that (5) has a weak solution on $Q_T = (0, T) \times \Omega$ for every set of initial concentrations such that $u_{i,0} \in L^2(\Omega; \mathbb{R}_+)$. This is interesting in itself, since the right-hand side is allowed to have quadratic nonlinearities and there is no restriction on the spatial dimension. Let us emphasize that this existence result cannot be deduced from standard approaches on semilinear problems, even for bounded initial data and even for $n = 2$ (the situation is different from the one described in (4)). In fact, the proof makes use of a recent L^2 -technique for such RD-systems with quadratic nonlinearities and control of mass; cf. [21], [22], [14]. The main result of this paper is the proof that, given a sequence $k \rightarrow \infty$, every sequence (u^k) of weak solutions to (5) has a subsequence converging

in $L^2(Q_T)$ to (v^+, v^-, w) , where (v, w) is a weak solution of (9). Of course the full sequence (u^k) would be convergent if the limit problem had unique weak solutions, but the latter is not known in general.

Let us note in passing that there are other standard situations in which instantaneous limits of reaction-diffusion systems can be obtained rigorously - at least for certain prototype models. These are cases with fast reversible reactions and systems with fast intermediates. Concerning the first class see [6], [7] and the references given there. For the latter case we refer to [4] and [9].

2. The case of finite reaction speed. For the proof of existence of a weak solution in this case, we do not need the specific structure of the fast irreversible reaction term. Indeed, we can absorb the irreversible reaction into f while keeping the assumptions (6), (7) and (8) intact. In case of finite reaction speed we therefore consider the RD-system

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(t, u) & (i = 1, \dots, n) \text{ on } (0, T) \times \Omega, \\ \partial_n u_i|_{\partial\Omega} = 0, \quad u_i|_{t=0} = u_{i,0}. \end{cases} \quad (10)$$

By a weak solution of (10) on $Q_T = (0, T) \times \Omega$, we mean that

$$u_i \in L^2(Q_T) \cap C([0, T]; L^1(\Omega)) \cap L^1((0, T); W^{1,1}(\Omega)), \quad (11)$$

and for all $\eta \in C^\infty([0, T] \times \bar{\Omega})$ with $\eta(T) = 0$ we have

$$\int_{Q_T} -\partial_t \eta u_i + d_i \nabla \eta \cdot \nabla u_i = \int_{\Omega} \eta(0) u_{i,0} + \int_{Q_T} f_i(t, u) \eta. \quad (12)$$

Note that for any weak solution of this type and under the given assumptions, the nonlinear terms $f_i(t, u)$ are in $L^1(Q_T)$ since all components u_i are in $L^2(Q_T)$. Since all components belong to $L^1(0, T; W^{1,1}(\Omega))$, it follows that $\nabla u_i \in L^1(Q_T)$. Therefore, the equation (12) does make sense. Note also that the homogeneous Neumann boundary conditions are incorporated in a weak sense due to the above choice of test functions.

Let us start with existence of weak solutions for finite reaction speed.

Theorem 2.1. *Assume that $f : [0, T] \times \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ satisfies (6), (7) and (8). Let $d_i > 0$ and $u_0 \in L^2(\Omega; \mathbb{R}_+^n)$ be given. Then (10) has a nonnegative weak solution.*

As explained in the introduction, this existence result does not follow from classical results on semilinear parabolic problems and provides only weak (but global) solutions. It could be deduced from the general global existence result stated in [20] for systems satisfying conditions like (7) together with nonlinearities bounded in $L^1(Q_T)$: we will see that this L^1 -bound indeed holds here, thanks to a priori L^2 -estimates described below. Details of this approach may be found in the appendix of [14].

We give here a more direct proof using more directly L^2 -estimates on the quantity $\langle u, e \rangle$, since the latter is also useful for the passage $k \rightarrow \infty$ later on. These L^2 -estimates have been recently introduced in [21], [22] and further developed in [14], [9].

For the present purpose, let us start with such an L^2 -estimate for the related RD-system

$$\begin{cases} \partial_t u - D\Delta u = g(t, x) & \text{on } Q_T, \\ \partial_n u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \end{cases} \quad (13)$$

with right-hand side in $L^1(Q_T; \mathbb{R}^n)$. It is well-known that (13) has a unique weak (=mild) solution on Q_T , for every $g \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$; cf. [10], [2] and Lemma 2.3 below. Here by a weak solution of (13) on Q_T , we mean that

$$u_i \in C([0, T]; L^1(\Omega)) \cap L^1((0, T); W^{1,1}(\Omega)), \quad (14)$$

and for all $\eta \in C^\infty([0, T] \times \bar{\Omega})$ with $\eta(T) = 0$ it holds that

$$\int_{Q_T} -\partial_t \eta u_i + d_i \nabla \eta \cdot \nabla u_i = \int_{\Omega} \eta(0) u_{i,0} + \int_{Q_T} g_i \eta. \quad (15)$$

Now for L^2 -initial values and if g allows for an L^2 -control of the total mass, we actually obtain the additional integrability $u \in L^2(Q_T; \mathbb{R}^n)$ for (componentwise) nonnegative solutions. In fact, more is true: if u^k are nonnegative solutions of (13) for right-hand sides g^k with L^2 -control of the total mass independent of k , then boundedness of (g^k) in $L^1(Q_T)$ implies relative compactness of (u^k) in $L^2(Q_T)$. This is contained in

Lemma 2.2. *Let $D = \text{diag}(d_1, \dots, d_n)$ with $d_i > 0$ and, for every $k \in \mathbb{N}$, let u^k be the weak solution of*

$$\begin{cases} \partial_t u^k - D\Delta u^k = g^k(t, x) & \text{on } Q_T, \\ \partial_n u^k|_{\partial\Omega} = 0, \quad u^k|_{t=0} = u_0^k, \end{cases} \quad (16)$$

where we assume

$$\begin{cases} u^k \geq 0, \quad u_0^k \rightarrow u_0 \text{ in } L^2(\Omega; \mathbb{R}^n), \\ (g^k) \text{ bounded in } L^1(Q_T, \mathbb{R}^n), \langle g^k, e \rangle \leq \phi_0, \text{ where } e \in \mathbb{R}^n, e \gg 0, \\ \phi_0 \in L^1(Q_T, \mathbb{R}^n), [(t, x) \rightarrow \Phi_0(t, x) = \int_0^t \phi_0(s, x) ds] \in L^2(Q_T, \mathbb{R}^n). \end{cases} \quad (17)$$

Then, the sequence (u^k) is relatively compact in $L^2(Q_T, \mathbb{R}^n)$.

For the proof of Lemma 2.2 we will employ the following compactness result which is more or less classical; for a proof see, e.g., [2] or [9].

Lemma 2.3. *Let $d > 0$. The mapping $(w_0, \Theta) \rightarrow w$, where w is the solution of*

$$w_t - d\Delta w = \Theta, \quad \partial_n w = 0 \text{ on } \partial\Omega, \quad w(0, \cdot) = w_0,$$

is compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1((0, T); W^{1,1}(\Omega))$.

Proof of Lemma 2.2. Using standard mollification, we may approximate u_0^k, g^k by regular functions $u_0^{k,l} \in C^\infty(\bar{\Omega}), g^{k,l} \in C^\infty(\bar{Q}_T)$ in $L^2(\Omega)$ and $L^1(Q_T)$, respectively, in such a way that $\langle g^{k,l}, e \rangle \leq \phi_0^l$ with $\phi_0^l \rightarrow \phi_0$ in $L^1(Q_T)$ and $\Phi_0^l(t, x) = \int_0^t \phi_0^l(s, x) ds$ converges to Φ_0 in $L^2(Q_T)$. It is sufficient to show that the sequence $(u^{k,l})_{k,l \geq 1}$ of solutions to (16) with data $u_0^{k,l}, g^{k,l}$ instead of u_0^k, g^k , is relatively compact in $L^2(Q_T, \mathbb{R}^n)$ (since the limit as $l \rightarrow \infty$ of $u^{k,l}$ is u^k , the relative compactness of $(u^k)_{k \geq 0}$ will follow).

Let us denote by (u^m) a subsequence of $(u^{k,l})$. By Lemma 2.3 we may assume, up to a subsequence, that u^m converges in $L^1(Q_T, \mathbb{R}^n)$ and a.e. on Q_T to some u .

Let us show that the convergence holds also in $L^2(Q_T, \mathbb{R}^n)$. This will imply the relative compactness of $(u^{k,l})$ in $L^2(Q_T, \mathbb{R}^n)$ and complete the proof of Lemma 2.2.

For this purpose, we introduce the solutions v^m of

$$\begin{cases} \partial_t v^m - D\Delta v^m = g^m + (\phi_0^m - \langle g^m, e \rangle) \frac{e}{|e|^2} & \text{on } Q_T, \\ \partial_n v^m|_{\partial\Omega} = 0, \quad v^m|_{t=0} = u_0^m. \end{cases} \quad (18)$$

Note that, since $\partial_t v^m - D\Delta v^m \geq g^m$, $v^m(0) = u^m(0)$, then

$$0 \leq u^m \leq v^m \quad (\text{componentwise}). \quad (19)$$

The right-hand side of (18) being bounded in $L^1(Q_T, \mathbb{R}^n)$, we may assume by Lemma 2.3 that, up to a subsequence, v^m converges in $L^1(Q_T, \mathbb{R}^n)$ and a.e. on Q_T to some $v \in L^1(Q_T, \mathbb{R}^n)$. We will show that v^m converges actually in $L^2(Q_T, \mathbb{R}^n)$. Thanks to (19) and to the a.e. convergence of u^m , the L^2 -convergence of u^m , up to a subsequence, will follow by the dominated convergence theorem.

We denote $W^m := \langle v^m, e \rangle$ and $Z^m := \langle Dv^m, e \rangle$. Then, taking inner product of the pde-system in (18) with e leads to

$$\int_{Q_T} -\partial_t \eta W^m + \int_{Q_T} \nabla \eta \cdot \nabla Z^m = \int_{\Omega} \eta(0) W^m(0) + \int_{Q_T} \eta \phi_0^m \quad (20)$$

for all $\eta \in C^\infty([0, T] \times \bar{\Omega})$ with $\eta(T) = 0$, and, by density, for all $\eta \in C^1([0, T] \times \bar{\Omega})$ with $\eta(T) = 0$. Thanks to the regularity of Z^m , we may choose $\eta = \int_t^T Z^m(s) ds$ in this relation which gives

$$\int_{Q_T} W^m Z^m + \frac{1}{2} \int_{\Omega} \left| \int_0^T \nabla Z^m(s) ds \right|^2 = \int_{Q_T} [W^m(0) + \Phi_0^m] Z^m. \quad (21)$$

Now note that

$$d_{\min} W^m \leq Z^m \leq d_{\max} W^m \quad \text{with } d_{\min} = \min_i d_i, \quad d_{\max} = \max_i d_i.$$

Hence (21) implies that Z^m, W^m are bounded in $L^2(Q_T, \mathbb{R}^n)$ and $\int_0^T \nabla Z^m(s) ds$ is bounded in $L^2(\Omega)$. Up to a subsequence, they are weakly convergent in these L^2 -spaces and their limits are, respectively,

$$W = \langle v, e \rangle, \quad Z = \langle Dv, e \rangle, \quad \int_0^T \nabla Z(s) ds.$$

In particular, $W, Z \in L^2(Q_T)$, $\int_0^T \nabla Z(s) ds \in L^2(\Omega)$.

Similarly, choosing $\eta = \int_t^\tau Z^m(s) ds$ in (20) with $\tau \in (0, T)$, the same computations lead to $\int_0^\tau \nabla Z(s) ds \in L^2(\Omega)$ with a bound depending only on T . By taking differences, $\int_\tau^T \nabla Z(s) ds \in L^2(\Omega)$. For future reference, we note that

$$W, Z \in L^2(Q_T) \quad \text{and} \quad \int_t^T \nabla Z(s) ds \in L^\infty(0, T; L^2(\Omega)) \quad \forall t \in (0, T). \quad (22)$$

Let us now pass to the limit in (21). Using the a.e. convergence of W^m, Z^m together with Fatou's Lemma, the weak L^2 -convergence of $\int_0^T \nabla Z^m(s) ds$ and the strong L^2 -convergence of $(W^m(0), \Phi_0^m)$, we obtain

$$\int_{Q_T} WZ + \frac{1}{2} \int_{\Omega} \left| \int_0^T \nabla Z(s) ds \right|^2 \leq \int_{Q_T} [W(0) + \Phi_0] Z. \quad (23)$$

The main step of the proof is to check that equality, rather than inequality, holds in (23). To obtain this equality, we first notice that, by density, (20) remains valid for all $\eta \in W^{1,2}(Q_T)$ such that $\eta(T) = 0$. Due to (22) we may apply it to $\eta := \int_t^T Z(s)ds$ to the result

$$\int_{Q_T} Z W^m + \int_{Q_T} \nabla Z^m \cdot \nabla \int_t^T Z(s)ds = \int_{Q_T} [W^m(0) + \Phi_0^m] Z(s)ds. \quad (24)$$

On the other hand, passing to the limit in (20) gives that, for all $\eta \in C^1([0, T] \times \bar{\Omega})$ with $\eta(T) = 0$,

$$\int_{Q_T} -\partial_t \eta W + \int_{Q_T} \nabla \eta \cdot \nabla Z = \int_{\Omega} \eta(0)W(0) + \int_{Q_T} \eta \phi_0.$$

We may choose $\eta = \int_t^T Z^m(s)ds$ in this equation, so that

$$\int_{Q_T} Z^m W + \int_{Q_T} \nabla Z \cdot \nabla \int_t^T Z^m(s)ds = \int_{Q_T} [W(0) + \Phi_0] Z^m(s)ds. \quad (25)$$

Now, we add up (24) and (25) and use the identity

$$\begin{aligned} & \int_{Q_T} \left(\nabla Z^m \cdot \nabla \int_t^T Z(s)ds + \nabla Z \cdot \nabla \int_t^T Z^m(s)ds \right) \\ &= \int_{\Omega} \left(\int_0^T \nabla Z^m(s)ds \right) \cdot \left(\int_0^T \nabla Z(s)ds \right). \end{aligned}$$

Then, passing to the limit in the sum of (24) and (25), using only weak L^2 -convergences, we arrive at

$$2 \int_{Q_T} ZW + \int_{\Omega} \left| \int_0^T \nabla Z(s)ds \right|^2 = 2 \int_{Q_T} [W(0) + \Phi_0] Z(s)ds,$$

which exactly means equality in (23).

From this equality we directly infer that

$$\lim_{m \rightarrow \infty} \int_{Q_T} W^m Z^m = \int_{Q_T} W Z \quad (26)$$

as well as

$$\lim_{m \rightarrow \infty} \int_{\Omega} \left| \int_0^T \nabla Z^m(s)ds \right|^2 = \int_{\Omega} \left| \int_0^T \nabla Z(s)ds \right|^2. \quad (27)$$

Equation (26) implies the strong convergence of W^m and Z^m in $L^2(Q_T)$. Indeed, we already know that $a^m := Z^m/W^m$ converges to $a := Z/W$ a.e. on Q_T . Moreover,

$$0 < d_{\min} \leq a^m \leq d_{\max} \quad \text{a.e. on } Q_T$$

shows that

$$\lim_{m \rightarrow \infty} \int_{Q_T} a^m (W^m - W)^2 = \lim_{m \rightarrow \infty} \int_{Q_T} (Z^m W^m - 2Z^m W + a^m W^2) = 0.$$

Hence indeed $W^m \rightarrow W$ in $L^2(Q_T)$ and then $0 \leq v^m \leq W^m$, together with the a.e. convergence of v^m , yields $v^m \rightarrow v$ in $L^2(Q_T)$ which ends the proof. \square

Remark 1. The proof of Lemma 2.2 includes the following estimate for any non-negative weak solution u of (13) under the assumptions $d_i > 0$, $u_0 \in L^2(\Omega; \mathbb{R}_+^n)$ and $g \in L^1(Q_T)$ such that $\langle g, e \rangle \leq \phi_0$ with $e \gg 0$ for some $\phi_0 \in L^1(Q_T)$, where $\Phi_0(t, x) = \int_0^t \phi(s, x) ds$ belongs to $L^2(Q_T)$. In this situation, $W := \langle u, e \rangle$ and $Z := \langle Du, e \rangle$ satisfy

$$\int_{Q_T} WZ + \frac{1}{2} \int_{\Omega} \left| \int_0^T \nabla Z(s) ds \right|^2 \leq \int_{\Omega} [W(0) + \Phi_0] Z, \quad (28)$$

and equality holds if $\langle g, e \rangle = \phi_0$.

Proof of Theorem 2.1. Notice first that by rewriting the system (5) in terms of the new variable $\tilde{u}_i = e^{-Lt} u_i$ with L from (7), we can absorb the term $\langle u, e \rangle$ and may therefore assume that f satisfies

$$\langle f(t, u), e \rangle \leq \phi \quad \text{with some } e \gg 0, \phi \in L^2(Q_T) \quad (29)$$

instead of (7).

Let $\pi_R : \mathbb{R}^n \rightarrow [0, R]^n$ be the metric projection onto $[0, R]^n$ with $(\pi_R y)_i = \max\{0, \min\{y_i, R\}\}$, define $f^R(t, y) := f(t, \pi_R y)$ and consider the approximating RD-system

$$\begin{cases} \partial_t u - D\Delta u = f^R(t, u) & \text{on } (0, T) \times \Omega, \\ \partial_n u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0^R := \pi_R(u_0). \end{cases} \quad (30)$$

Evidently, f^R has the same properties as f and, in addition, it is bounded and Lipschitz continuous in the second variable. Hence, by well-known theory (see, e.g., [1] or [18]), the system (30) has a unique strong (in the L^p -sense for all $p \geq 1$) solution u^R which is in fact bounded on Q_T . From (28) in the remark behind the proof of Lemma 2.2 it follows in particular that $(u^R)_{R>0}$ is bounded in $L^2(Q_T)$. Hence, due to (6), the right-hand sides $f^R(\cdot, u^R)$ are uniformly bounded in $L^1(Q_T)$ and therefore Lemma 2.2 applies. Consequently, up to a subsequence, we may assume $u^R \rightarrow u$ in $L^2(Q_T)$ and a.e. on Q_T and then also $f^R(\cdot, u^R) \rightarrow f(\cdot, u)$ in $L^1(Q_T)$. The latter implies that u is a weak solution for the right-hand side $f(\cdot, u)$, hence a weak solution of (10). \square

3. Convergence to the instantaneous limit. Instead of system (5), it will be convenient to consider the slightly more general case

$$\begin{cases} \partial_t u - D\Delta u = -k r(t, u) \nu + f(t, u) & \text{on } (0, +\infty) \times \Omega, \\ \partial_n u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0, \end{cases} \quad (31)$$

where $\nu = (1, 1, 0, \dots, 0)$ and $r : [0, T] \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ is jointly continuous, locally Lipschitz in the second variable with $r(t, y) \geq 0$ and $r(t, y) = 0$ if and only if $y_1 y_2 = 0$. Note that the more general case $\nu = (\alpha, \beta, 0, \dots, 0)$ with $\alpha, \beta > 0$ can be reduced to the former one by absorbing these stoichiometric factors into u_1 and u_2 and redefining r and f without changing their properties.

Now the main result of this paper reads as

Theorem 3.1. *Assume that $f : [0, T] \times \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ satisfies (6), (7) and (8). Let $r : [0, T] \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ be jointly continuous and locally Lipschitz continuous in the second variable with at most quadratic growth, $r(t, y) \geq 0$ and $r(t, y) = 0$ if and only if $y_1 y_2 = 0$. Let $d_i > 0$ and $u_0 \in L^2(\Omega; \mathbb{R}_+^n)$. For $k > 0$ let u^k be a nonnegative weak solution of (31) which exists due to Theorem 2.1. Then, given any sequence*

$k_l \rightarrow \infty$, the sequence of weak solutions (u^{k_l}) has a subsequence (u^{m_l}) such that $u^{m_l} \rightarrow (v^+, v^-, w)$ in $L^2(Q_T)$, where (v, w) is a weak solution of the limit system (9), i.e.

$$\left\{ \begin{array}{l} \partial_t v - \Delta \phi(v) = g(t, v, w) \\ \partial_t w - D \Delta w = h(t, v, w) \end{array} \right\} \text{ on } (0, +\infty) \times \Omega,$$

$$\partial_n \phi(v)|_{\partial \Omega} = 0, \partial_n w|_{\partial \Omega} = 0, \quad v|_{t=0} = v_0, w|_{t=0} = w_0$$

with $\phi(s) = d_1 s$ for $s \geq 0$, $\phi(s) = d_2 s$ otherwise, $D = \text{diag}(d_3, \dots, d_n)$,

and

$$g(t, v, w) = f_1(t, v^+, v^-, w) - f_2(t, v^+, v^-, w)$$

$$h(t, v, w) = (f_3(t, v^+, v^-, w), \dots, f_n(t, v^+, v^-, w)).$$

The initial value for (9) is $v_0 = u_{0,1} - u_{0,2}$, $w_0 = (u_{0,3}, \dots, u_{0,n})$.

Proof. Given $k_l \rightarrow \infty$ and a sequence of weak solutions (u^{k_l}) of (31) for fixed initial value u_0 , let $W^l := \langle u^{k_l}, e \rangle$ and $Z^l := \langle Du^{k_l}, e \rangle$ with $e \gg 0$ from (7). By our definition of a weak solution of (31) it holds that $u^{k_l} \in L^2(Q_T; \mathbb{R}^n)$, hence $f(\cdot, u^{k_l}) \in L^1(Q_T; \mathbb{R}^n)$ by (6). From (28) in the remark behind the proof of Lemma 2.2 we have

$$\int_{Q_T} W^l Z^l + \frac{1}{2} \int_{\Omega} \left| \int_0^T \nabla Z^l(s) ds \right|^2 \leq \int_{Q_T} [W(0) + \Phi_0] Z^l \quad (32)$$

with fixed $\Phi_0 \in L^2(Q_T)$; in particular, W^l, Z^l and u^l are uniformly bounded in $L^2(Q_T)$, as is $\int_0^T \nabla Z^l(s) ds$ in $L^2(\Omega)$. Thanks to (6), this implies that $f(\cdot, u^{k_l})$ is bounded in $L^1(Q_T)$ independently of l . Integration of the first equation of (31) over Q_T , and using nonnegativity of u^{k_l} shows that $k_l r(\cdot, u^{k_l})$ is also bounded in $L^1(Q_T)$. Therefore, Lemma 2.3 applies and yields a subsequence (u^{m_l}) such that

$$u^{m_l} \rightarrow u \text{ in } L^1(Q_T), \quad \nabla u^{m_l} \rightarrow \nabla u \text{ in } L^1(Q_T; \mathbb{R}^n)$$

for some $u \in L^1((0, T); W^{1,1}(\Omega))$. We may also assume $u^{m_l} \rightarrow u$ and $f(\cdot, u^{m_l}) \rightarrow f(\cdot, u)$ a.e. on Q_T . Moreover, Lemma 2.2 shows that, up to a subsequence,

$$u^{m_l} \rightarrow u \text{ in } L^2(Q_T; \mathbb{R}^n), \quad f(\cdot, u^{m_l}) \rightarrow f(\cdot, u) \text{ in } L^1(Q_T; \mathbb{R}^n).$$

To see that (v, w) with $v = u_1 - u_2$, $w = (u_3, \dots, u_n)$ is a weak solution of (9), let $\eta \in C^\infty([0, T] \times \bar{\Omega})$ with $\eta(T) = 0$ and note that it suffices to consider the first component v ; the weak version of the other equations for the w_i is then clear. Taking the difference of the weak versions of the first two differential equations in (31), we obtain

$$\begin{aligned} & \int_{Q_T} -\partial_t \eta (u_1^{m_l} - u_2^{m_l}) + \nabla \eta \cdot \nabla (d_1 u_1^{m_l} - d_2 u_2^{m_l}) \\ &= \int_{\Omega} \eta(0) (u_{1,0} - u_{2,0}) + \int_{Q_T} (f_1(t, u^{m_l}) - f_2(t, u^{m_l})) \eta. \end{aligned}$$

In the limit as $l \rightarrow \infty$, this yields

$$\int_{Q_T} -\partial_t \eta v + \nabla \eta \cdot \nabla (d_1 u_1 - d_2 u_2) = \int_{\Omega} \eta(0) v_0 + \int_{Q_T} (f_1(t, u_1) - f_2(t, u_2)) \eta.$$

Since $k_l \int_{Q_T} r(\cdot, u^{k_l})$ is bounded, $\int_{Q_T} r(\cdot, u^{k_l}) \rightarrow 0$ so that we also have $r(\cdot, u) = 0$; hence $u_1 u_2 = 0$ a.e. on Q_T by the assumptions on r . Thus $u_1 = v^+$, $u_2 = v^-$ and

$$\int_{Q_T} -\partial_t \eta v + \nabla \eta \cdot \nabla \phi(v) = \int_{\Omega} \eta(0) v_0 + \int_{Q_T} g(t, v, w) \eta$$

holds for all test functions. Therefore, with the corresponding equations for w , the pair (v, w) is a weak solution of the limit system (9) and the stated $L^2(Q_T)$ -convergence is obvious. \square

Remark 2. The solutions u^k of the original RD-system belong to $C([0, T], X)$ with $X = L^1(\Omega; \mathbb{R}^n)$. In the general case, we cannot expect convergence in the latter space, since a boundary layer builds up near $t = 0$. But, given a sequence (u^{k_l}) for $k_l \rightarrow \infty$, Theorem 3.1 yields a subsequence (u^{m_l}) such that $u^{m_l}(\tau, \cdot) \rightarrow u(\tau, \cdot)$ in $L^2(\Omega)$ for a.e. $\tau \in (0, T)$ with $u = (v^+, v^-, w)$. Actually, using the approach in [3] and [5], it is then possible to prove that $u^{k_l} \rightarrow u$ in $C([\tau, T], X)$ for any $\tau > 0$.

Let us close with an example which appears in concrete applications (see [8]) and where convergence of the full sequence (u^k) as $k \rightarrow \infty$ can be obtained in the realistic case of bounded initial concentrations.

Example. Consider a chemically reacting system with the two parallel reactions $A + B \rightarrow P$ and $A + C \rightleftharpoons Q$, which are competing for A . We assume that the first (irreversible) reaction is considerably faster than the second one. Then, assuming mass action kinetics and under an appropriate scaling of time, the original system for finite reaction speed reads as

$$\left\{ \begin{array}{l} \partial_t c_A - d_A \Delta c_A = -k c_A c_B - c_A c_C + \kappa c_Q \\ \partial_t c_B - d_B \Delta c_B = -k c_A c_B \\ \partial_t c_C - d_C \Delta c_C = -c_A c_C + \kappa c_Q \\ \partial_t c_Q - d_Q \Delta c_Q = c_A c_C - \kappa c_Q \end{array} \right\} \text{ on } (0, +\infty) \times \Omega, \quad (33)$$

$$\left. \begin{array}{l} \partial_n c_A|_{\partial\Omega} = \partial_n c_B|_{\partial\Omega} = \partial_n c_C|_{\partial\Omega} = \partial_n c_Q|_{\partial\Omega} = 0, \\ c_A|_{t=0} = c_{A,0}, c_B|_{t=0} = c_{B,0}, c_C|_{t=0} = c_{C,0}, c_Q|_{t=0} = c_{Q,0}. \end{array} \right\}$$

Writing c for $c_A - c_B$, the limit system is given as

$$\left\{ \begin{array}{l} \partial_t c - \Delta \phi(c) = -c^+ c_C + \kappa c_Q \\ \partial_t c_C - d_C \Delta c_C = -c^+ c_C + \kappa c_Q \\ \partial_t c_Q - d_Q \Delta c_Q = c^+ c_C - \kappa c_Q \end{array} \right\} \text{ on } (0, +\infty) \times \Omega, \quad (34)$$

$$\left. \begin{array}{l} \partial_n \phi(c)|_{\partial\Omega} = \partial_n c_C|_{\partial\Omega} = \partial_n c_Q|_{\partial\Omega} = 0, \\ c|_{t=0} = c_{A,0} - c_{B,0}, c_C|_{t=0} = c_{C,0}, c_Q|_{t=0} = c_{Q,0} \end{array} \right\}$$

with $\phi(s) = d_A s$ for $s \geq 0$ and $\phi(s) = d_B s$ otherwise.

Now assume $c_{i,0} \in L^\infty(\Omega)$ with $c_{i,0} \geq 0$, let $(k_l) \subset \mathbb{R}_+$ be any sequence with $k_l \rightarrow \infty$ and $(c_A^{k_l}, c_B^{k_l}, c_C^{k_l}, c_Q^{k_l})$ be a weak solution of (33). By Theorem 3.1 we find a subsequence such that

$$(c_A^{m_l}, c_B^{m_l}, c_C^{m_l}, c_Q^{m_l}) \rightarrow (c^+, c^-, c_C, c_Q) \text{ in } L^2(Q_T),$$

where (c, c_C, c_Q) is a weak solution of (34). But for L^∞ -initial data, it is easy to check by the techniques from [19], [21] (cf. also [9]) that all c_i^k , $i = A, B, C, Q$ are uniformly bounded in $L^\infty(Q_T)$, where the bounds are independent of $k > 0$. Therefore, the limit (c^+, c^-, c_C, c_Q) corresponds to a *bounded* weak solution of (34), and the latter are unique. Consequently, the limit of any convergent subsequence is always the same and, hence, the full sequence $(c_A^{k_l}, c_B^{k_l}, c_C^{k_l}, c_Q^{k_l})$ converges to (c^+, c^-, c_C, c_Q) in $L^2(Q_T)$, and even in any $L^p(Q_T)$ for $p < +\infty$, thanks to the L^∞ -bound.

REFERENCES

- [1] H. Amann, *Global existence for semilinear parabolic problems*, J. Reine Angew. Math. **360** (1985), 47–83.
- [2] P. Baras, M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, (French, English summary), Applicable Analysis **18** (1984), 111–149.
- [3] Ph. Bénilan, M.G. Crandall, A. Pazy, “Nonlinear Evolution Equations in Banach Spaces”, preprint book.
- [4] M. Bisi, F. Conforto, L. Desvillettes, *Quasi-steady-state approximation for reaction-diffusion equations*, Bull. Inst. Math., Acad. Sin. (N.S.) **2** (2007), 823–850.
- [5] D. Bothe, *The instantaneous limit of a reaction-diffusion system*, in “Evolution Equations and Their Applications in Physical and Life Sciences” (eds. G. Lumer, L. Weis), Marcel Dekker, Lect. Notes Pure Appl. Math. **215** (2000), 215–224.
- [6] D. Bothe, *Instantaneous limits of reversible chemical reactions in presence of macroscopic convection*, J. Differ. Equations **193** (2003), 27–48.
- [7] D. Bothe, D. Hilhorst, *A reaction-diffusion system with fast reversible reaction*, J. Math. Anal. Appl. **286** (2003), 125–135.
- [8] D. Bothe, A. Lojewski, H.-J. Warnecke, *Computational analysis of an instantaneous chemical reaction in a T-microreactor*, AIChE J. (to appear).
- [9] D. Bothe, M. Pierre, *Quasi-steady-state approximation for a reaction-diffusion system with fast intermediate*, submitted.
- [10] H. Brezis, W.A. Strauss, *Semi-linear second-order elliptic equations in L^1* , J. Math. Soc. Japan **25** (1973), 565–590.
- [11] J.R. Cannon, C.D. Hill, *On the movement of a chemical reaction interface*, Indiana Univ. Math. J. **20** (1970), 429–453.
- [12] J.R. Cannon, A. Fasano, *Boundary value multidimensional problems in fast chemical reactions*, Archive Rat. Mech. Anal. **53** (1973), 1–13.
- [13] E.N. Dancer, D. Hilhorst, M. Mimura, L.A. Peletier, *Spatial segregation limit of a competition-diffusion system*, European J. Appl. Math. **10** (1999), 97–115.
- [14] L. Desvillettes, K. Fellner, M. Pierre, J. Vovelle, *About Global Existence for Quadratic Systems of Reaction-Diffusion*, Advanced Nonlinear Studies **7** (2007), 491–511.
- [15] P. Erdi, J. Toth, “Mathematical Models of Chemical Reactions”, Manchester Univ. Press, Manchester 1989.
- [16] L.C. Evans, *A convergence theorem for a chemical diffusion-reaction system*, Houston J. Math. **6** (1980), 259–267.
- [17] D. Hilhorst, M. Mimura, H. Ninomiya, *Fast reaction limit of competition-diffusion systems*, in “Handbook of Differential Equations” (eds. C.M. Dafermos and M. Pokorný), Elsevier, Evolution Equations **5** (2009).
- [18] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser Verlag, Basel, Progress in Nonlinear Differential Equations and Their Applications (1995).
- [19] R.H. Martin and M. Pierre, *Nonlinear reaction-diffusion systems*, in “Nonlinear Equations in the Applied Sciences” (eds. W.F. Ames and C. Rogers), Acad. Press, New York, Math. Sci. Eng. **185** (1991).
- [20] M. Pierre, *Weak solutions and supersolutions in L^1 for reaction-diffusion systems*, J. Evol. Equ. **3** (2003), 153–168.
- [21] M. Pierre, D. Schmitt, *Blowup in reaction-diffusion systems with dissipation of mass*, SIAM J. Math. Anal. **28** (1997), 259–269.
- [22] M. Pierre, D. Schmitt, *Blowup in reaction-diffusion systems with dissipation of mass*, SIAM Rev. **42** (2000), 93–106 (electronic).
- [23] Y. Tonegawa, *On the regularity of a chemical reaction interface*, Comm. PDEs **23** (1998), 1181–1207.
- [24] J.L. Vazquez, “The Porous Medium Equation, Mathematical Theory”, Clarendon-Press, Oxford University Press, Oxford (2007).

E-mail address: bothe@csi.tu-darmstadt.de

E-mail address: michel.pierre@bretagne.ens-cachan.fr