Ecole Normale Supérieure de Cachan

Ecole Doctorale Sciences Pratiques

Habilitation à diriger des recherches

Spécialité : Mathématiques appliquées

Karine Beauchard

Analyse et contrôle de quelques equations aux dérivées partielles.

Soutenue le 9 novembre 2010

-Jury-

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	Andrei Fursikov	
	Jean-Pierre Puel	*Rapporteur*
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	Enrique Zuazua	

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Je suis extrêmement honorée que Nicolas Burq, Piermarco Cannarsa et Jean-Pierre Puel aient pris le temps d'étudier ce manuscrit et de rédiger un rapport. Tous ont accepté spontanément ce travail et je leur en suis reconnaissante.

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Je tiens à exprimer ma gratitude à Laurent Desvillettes pour avoir bien voulu être mon garant.

Je remercie tous les chercheurs avec qui j'ai eu la chance de collaborer et qui n'ont pas encore été cités : François Alouges, Yacine Chitour, Djalil Khateb, Camille Laurent, Ruixing Long, Mazyar Mirrahimi, Vahagn Nersesyan, Paulo Periera Da Silva, Mario Sigalotti. Je remercie également tous les participants du groupe de travail de l'ANR-CQUID, du groupe de travail 'contrôle' de Paris 6, et du GDRE 'contrôle des EDP' : ces rencontres mathématiques sont très stimulantes.

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Table des matières

I Présentation générale 11

1 Curriculum Vitae détaillé 13

1.1 Curriculum Vitae . . . 14
1.2 Publications et communications scientifiques . . . 14
1.2.1 Liste de publications . . . 14
1.2.2 Exposés . . . 15
1.3 Encadrement scientifique et enseignement . . . 17
1.3.1 Encadrement d'étudiants . . . 17
1.3.2 Enseignement et diffusion de la science . . . 17
1.4 Participation à la vie scientifique, responsabilités collectives . . . 19
1.5 Séjours de recherche à l'étranger . . . 19
1.6 Financements, réseaux . . . 20
1.7 Autres . . . 20

2 Résumé de mon activité de recherche 21

2.1 Introduction . . . 21
2.1.1 Présentation générale . . . 21
2.1.2 Rappel sur mon travail de thèse (**A1, A2, A3**) [22, 27, 28] . . . 22
2.1.3 Structure . . . 24
2.2 Contrôlabilité exacte de systèmes bilinéaires **(A4, A5, A12, A13)** [23, 24, 31, 25] . . . 25
2.3 Stabilisation d'équations de Schrödinger **(A6, A15)** [32, 33] . . . 28
2.4 Généricité et contrôle spectral pour une équation de Schrödinger linéaire **(A9)** [26] . . . 30
2.5 Contrôle et stabilisation de l'équation de Bloch **(A10, A14)** [29, 30] . . . 31
2.6 Systèmes hyperboliques **(A11)** [35] . . . 33
2.7 Contrôlabilité de l'équation de Kolmogorov **(A8)** [34] . . . 35
2.8 Contrôle en micromagnétisme **(A7,P1)** [10, 11] . . . 37

II Activité de recherche détaillée 41

3 Exact Controllability of bilinear systems with discrete spectrum (A4, A5, A12, A13) [23, 24, 31, 25] 45

3.1 Context, review of previous results . . . 45
3.1.1 Controllability of finite dimensional bilinear systems : the Lie rank condition . . . 46
3.1.2 Iterated Lie brackets in infinite dimension . . . 48

3.1.3 Ball, Marsden and Slemrod's negative result . . . 52
3.1.4 First positive exact controllability results in infinite dimension . . . 53
3.1.5 Other results about bilinear systems . . . 55
3.2 Technics used in this chapter . . . 56
3.2.1 Linearization principle . . . 56
3.2.2 Moment method . . . 56
3.2.3 Nash-Moser theorem . . . 56
3.2.4 Coron's return method . . . 56
3.2.5 Power series expansions . . . 57
3.3 Exact controllability of a particle in a variable domain [23] **(A4)** . . . 57
3.3.1 Equation and result . . . 57
3.3.2 Interest . . . 58
3.3.3 Technics and remarks . . . 59
3.4 Exact controllability of a bilinear beam equation [24] **(A5)** . . . 62
3.4.1 Equation and result . . . 62
3.4.2 Technics, remarks . . . 63
3.5 Exact controllability of a bilinear wave equation [25] **(A13)** . . . 63
3.5.1 Equation and result . . . 63
3.5.2 Technics and interest . . . 66
3.6 A simpler proof [31] **(A12)** . . . 66
3.6.1 Equation and result . . . 66
3.6.2 Technics and interest . . . 68
3.6.3 Additional results . . . 68
3.7 Conclusion, open problems, perspectives . . . 71
3.7.1 Schrödinger equations on variable domains . . . 71
3.7.2 Exact controllability of the bilinear Schrödinger equation on multidimensional domains . . . 72
3.7.3 Simultaneous controllability of N Schrödinger equations . . . 74

4 Stabilization of Schrödinger équations with discrete spectrum (A6, A15) [32, 33] 77
4.1 Introduction . . . 77
4.2 Approximate stabilization of a 1D bilinear Schrödinger equation [32] **(A6)** . 78
4.2.1 Equation and result . . . 78
4.2.2 Heuristic . . . 80
4.3 Weak stabilization of bilinear Schrödinger equations [33] **(A15)** . . . 82
4.3.1 Equation, heuristic . . . 82
4.3.2 Main result . . . 83
4.4 Conclusion, open problems, perspectives . . . 84
4.4.1 Weak stabilization on unbounded domains . . . 84
4.4.2 Strong stabilization of bilinear Schrödinger equations . . . 84
4.4.3 Simultaneous stabilization of N Schrödinger equations with one control 84
4.4.4 Stabilization of nonlinear Schrödinger equation . . . 85

5 Genericity and spectral control for a linear Schrödinger equation (A9) [26] 87
5.1 Introduction . . . 87
5.2 Results . . . 88
5.3 Conclusion, open problems, perspectives . . . 91

6 Control and stabilization of the Bloch equation, a bilinear system with continuous spectrum (A10, A14) [29, 30] 93
6.1 Introduction 93
6.2 Controllability **(A10)** [29] 94
6.2.1 Non exact controllability with a priori bounded controls 95
6.2.2 Approximate controllability with unbounded controls 96
6.2.3 Asymptotic exact controllability with unbounded controls 97
6.3 Stabilization **(A14)** [30] 99
6.3.1 The impulse-train control to reduce dispersion 99
6.3.2 Heuristics of the Lyapunov-like control 99
6.3.3 Convergence result 100
6.3.4 No global stabilization 101
6.4 Conclusion, open problems, perspectives 101

7 Hyperbolic systems (A11) [35] 103
7.1 Introduction 103
7.2 Asymptotic behavior of linear systems 104
7.2.1 Lyapunov function and explicit decay rates 104
7.2.2 L^2-stability and non dissipated solutions 106
7.2.3 Decomposition of the solutions 106
7.3 Improvements for nonlinear systems 109
7.3.1 Size of the neighborhood for global existence under (SK) 110
7.3.2 First application : global existence without (SK) 111
7.3.3 Second application : uniform well-posedness 113
7.4 Conclusion, open problems, perspectives 113
7.4.1 Complete classification of the asymptotic behavior of linear systems . 113
7.4.2 Nonlinear systems 114
7.4.3 Lyapunov functions 114

8 Controllability of the Kolmogorov equation (A8) [34] 117
8.1 Introduction 117
8.2 Results 119
8.3 Technics 120
8.4 Conclusion, open problems, perspectives 121

9 Control and micromagnetism (A7,P1) [10, 11] 123
9.1 Introduction 123
9.2 A Landau-Lifschitz ODE 125
9.3 PDE weak solutions 127
9.3.1 Existence of 3D global weak solutions 127
9.3.2 Convergence of weak solutions to ODE solutions when the size of the domain goes to zero 128
9.4 Global PDE smooth solutions 129
9.5 Exponential stabilization of uniform magnetizations on ellipsoidal domains . 131
9.6 Magnetization switching on ellipsoidal domains : PDE study. 132
9.7 Conclusion, open problems, perspectives 133

Ce manuscrit tend à présenter mon parcours dans le monde de la recherche depuis ma soutenance de thèse en décembre 2005.

La première partie résume ce parcours : les Chapitres 1 et 2 présentent mon CV détaillé et une description succinte de mes travaux.

La seconde partie du manuscrit (Chapitres 3 à 9) présente de manière plus précise mon activité de recherche (résumée au Chapitre 2). Cette seconde partie est rédigée en anglais, pour faciliter la compréhension d'eventuels lecteurs non francophones.

Première partie

Présentation générale

Chapitre 1

Curriculum Vitae détaillé

1.1 Curriculum Vitae

Karine Beauchard
née le 27/11/1978 à Romorantin (41), nationalité française, mariée, un enfant.

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Carrière et Formation :

sept 2010-	**Professeur chargé de cours** (exercice incomplet) au département de mathématiques de l'**Ecole Polytechnique.**
sept 2006-	**CR2 au CNRS**, membre du CMLA (ENS Cachan).
2005-2006	**Agrégée préparatrice** au département de mathématiques de l'ENS Cachan, membre du CMLA.
2002-2005	**Thèse** intitulée "Contribution à l'étude de la contrôlabilité et de la stabilisation de l'équation de Schrödinger ", sous la direction de **Jean-Michel Coron**, au Laboratoire Mathématique de l'Université d'**Orsay**. Rapporteurs : Gilles Lebeau, Enrique Zuazua. Jury : Jean-Pierre Puel, Jean-Pierre Raymond, Pierre Rouchon, Gabriel Turinici.
1999-2003	Scolarité à l'**ENS Cachan** : reçue 6e à l'agrégation de mathématiques en 2002, DEA d'analyse numérique à l'Université Paris VI.

Distinction

Prix Peccot 2008, décerné par le Collège de France, pour mes résultats sur la contrôlabilité d'équations de Schrödinger.

1.2 Publications et communications scientifiques

1.2.1 Liste de publications

Mes publications sont disponibles au format pdf sur ma page web.
http ://www.cmla.ens-cachan.fr/Membres/beauchard/publications.html

- **(A1)** K. Beauchard, *Local controllability of a 1D Schrödinger equation*, J. Math. Pures Appl., 84 :851-956, July 2005.
- **(A2)** K. Beauchard, J.-M. Coron, *Controllability of a quantum particle in a moving potential well*, J. of Functional Analysis, 232 (2006) p. 328-389.
- **(A3)** K. Beauchard, J.-M. Coron, M. Mirrahimi, P. Rouchon, *Implicit Lyapunov control of finite dimensional Schrödinger equations*, Systems and Control Letters, 56 : 388-395, May 2007.
- **(A4)** K. Beauchard, *Controllability of a quantum particule in a 1D variable domain*, ESAIM :COCV, volume 14, number 1, (2008) p. 105-147.

- **(A5)** K. Beauchard, *Local controllability of a 1D beam equation*, SIAM J. Control Optim., Volume 47, Issue 3, pp. 1219-1273 (2008).
- **(A6)** K. Beauchard and Mazyar Mirrahimi, *Practical stabilization of a quantum particle in a one-dimensional infinite square potential well*, SIAM J. Control Optim., 48 (2009), no. 2, p. 1179-1205.
- **(A7)** F. Alouges and K. Beauchard, *Magnetization switching on small ferromagnetic ellipsoidal samples*, ESAIM :COCV, 15 (2009), p. 676-711.
- **(A8)** K. Beauchard and E. Zuazua, *Some controllability results for the Kolmogorov equation*, Ann. I. H. Poincaré-AN, 26 (2009), p. 1793-1815.
- **(A9)** K. Beauchard, Y. Chitour, D. Kateb and R. Long, *Spectral controllability for 2D and 3D linear Schrödinger equations*, J. of Functional Analysis, vol. 256, p. 3916-3976, june, 2009.
- **(A10)** K. Beauchard, J.-M. Coron and P. Rouchon, *Controllability issues for continuous-spectrum systems and ensemble controllability of Bloch equations*, Communications in Mathematical Physics, volume 296, Number 2, June 2010, p.525-557.
- **(A11)** K. Beauchard and E. Zuazua, *Large time asymptotics for partially dissipative hyperbolic systems*, à paraître dans Arch. Rational Mech. Anal. DOI 10.1007/s00205-010-0321-y (51 pages).
- **(A12)** K. Beauchard and C. Laurent, *Local controllability of linear and nonlinear Schrödinger equations with bilinear control*, J. Math. Pures Appl., Volume 94, Issue 5, November 2010, p. 520-554.
- **(A13)** K. Beauchard, *Local controllability and non controllability of a 1D wave equation* à paraître dans Journal of Differential Equations (32 pages).
- **(A15)** K. Beauchard and V. Nersesyan, *Semi-global weak stabilization of bilinear Schrödinger equations*, CRAS, Volume 348, Issues 19-20, October 2010, p. 1073-1078.

Articles soumis :

- **(A14)** K. Beauchard, P. Pereira da Silva and P. Rouchon, *Stabilization of an ensemble of half spin system.* (21 pages)

Proceeding :

- **(P1)** F. Alouges, K. Beauchard and M. Sigalotti, *Magnetization switching in small ferromagnetic ellipsoidal samples*, Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, 2009.

1.2.2 Exposés

Conférences invitées dans des congrès internationaux :

1. septembre 2010, '8th IFAC Symposium on Nonlinear Control Systems', [Italie],
2. juillet 2010, Workshop 'Control of Partial Differential Equations' [CIME, Italie]
3. juin 2010, Workshop on control and inverse problems' [Besancon],
4. janvier 2010, GDR/GDRE 'Contrôle d'EDP' [CIRM],
5. septembre 2009, Physcon (international conference on Physics and Control) [Sicile],

6. mars 2009, congrès en l'honneur de Alain Haraux [Tunisie],
7. février 2009, Workshop 'Quantum Control', au Wolfgang Pauli Institute [utriche],
8. juin 2008, Congrès Control of Physical Systems and PDEs [IHP],
9. mars 2008, Congrès Franco-Taïwanais sur les EDP non linéaires, [CIRM],
10. décembre 2007, Worshop on PDE's, Numerical Analysis and Applications, [Portugal],
11. septembre 2007, Workshop on PDEs, optimal design and numerics [Espagne],
12. juin 2007, International Workshop on Analysis and Control of PDEs dédié à Jean-Pierre Puel [Pont à Mousson],
13. septembre 2006, Ecole d'été 'Equation dispersives' [Nice],
14. avril 2006, Congrès 'Problèmes Inverses, Contrôle et Optimisation de formes' [Nice],
15. septembre 2005, Workshop on PDEs, optimal design and numerics [Espagne],
16. juillet 2005, GDR EAPQ [Grenoble],
17. juin 2004, Talk au congrès en l'honneur de Haim Brézis [Paris].

<u>Séminaires de laboratoires :</u>

1. octobre 2010, Séminaire 'Applications des mathématiques', ENS Cachan-Bretagne,
2. décembre 2009, Séminaire commun d'analyse ENS/P6/P7,
3. novembre 2009, CMAP, Ecole Polytechnique,
4. novembre 2009, Université de Clermont Ferrand,
5. février 2009, LAGEP [Lyon],
6. avril 2008, Laboratoire Paul Painlevé [Lille],
7. février 2008, IECN [Nancy],
8. mai 2007, Collège de France,
9. mai 2007, UMR POems, INRIA Rocquencourt,
10. février 2007, Departamento de matematicas, UAM, Madrid,
11. février 2006, LACO [Limoges],
12. février 2006, Laboratoire Jean Leray [Nantes],
13. février 2006, ENS Lyon,
14. janvier 2006, Séminaire EDP analyse non linéaire, ENS ulm,
15. décembre 2005, X EDP du CMAT,
16. novembre 2005, LATP [Marseille],
17. septembre 2005, Journée de rentrée de l'équipe analyse numérique et EDP [Orsay]
18. février 2005, Université de Franche-Compté [Besançon],
19. janvier 2005, Séminaire des étudiants du MIP [Toulouse],
20. septembre 2004, SISSA [Trieste, Italie],
21. juin 2004, Institut Elie Cartan [Nancy],
22. avril 2004, Institut Girard Desargues [Lyon].

Groupes de travail, posters :

1. mai 2010, Groupe de travail de l'ANR C-QUID,
2. avril 2009, Groupe de travail 'Contrôle', Laboratoire JLL, Paris 6,
3. novembre 2009, Groupe de travail de l'ANR C-QUID,
4. avril 2009, Groupe de travail de l'ANR C-QUID,
5. octobre 2007, Groupe de travail de l'ANR C-QUID,
6. février 2006, Groupe de travail 'mécanique des fluides réels', CMLA, ENS Cachan,
7. mai 2005, Poster au congrès SMAI [Evian].

1.3 Encadrement scientifique et enseignement

1.3.1 Encadrement d'étudiants

- sept 2010- : encadrement de la **thèse** de Morgan Morancey (élève ENS Cachan, M2 analyse numérique de Paris 6),
- janvier-juin 2010 : encadrement à 100% de 4 élèves en **stage de L3** à l'ENS Cachan (stage long de recherche),
- 2007 : encadrement à 50%, du **stage de M2** de Ruixing Long (élève du M2 d'Orsay et de l'ENSTA) avec Yacine Chitour. Ce co-encadrement a contribué à la publication **(A9)** [26].

1.3.2 Enseignement et diffusion de la science

Cours au niveau recherche :

- mai 2009, Cours 'Contrôle d'équations de Schrödinger', **Ecole CIMPA**, Marrakech, (12h).
- janvier 2008, **Cours Peccot, Collège de France**, intitulé 'Contrôlabilité d'équations de Schrödinger' (8h),

Enseignements au niveau L3, M1, M2 :

2010-2011 : 88h de cours/TD :
- petites classes du cours 'Distributions, Analyse de Fourier et Systèmes Dynamiques' de François Golse et Raphaël Krikorian, au département de mathématiques de l'Ecole Polytechnique (74h),
- cours 'Dynamique, contrôle et estimation' au M2 MVA de l'ENS Cachan, en collaboration avec Pierre Rouchon (14h).

2009-2010 : 114h de cours + participation aux concours, au département de mathématiques de l'ENS Cachan :
- cours 'Dynamique et contrôle des systèmes nonlinéaires' au M2 MVA, en collaboration avec Pierre Rouchon (14h),
- cours/TD d'analyse complexe en L3 (30h),

- leçons d'analyse en préparation à l'agrégation (40h),
- encadrement de 4 stagiaires de L3 (30h),
- participation au concours d'entrée en 1ère année : conception de sujet, correction de copies, jury d'oraux.
- participation au concours d'entrée en 3ème année : jury d'oraux.

2006-2009 : environ 60h/an + participation aux concours, au département de mathématiques de l'ENS Cachan :
- oraux blancs d'analyse en préparation à l'agrégation,
- test de sujets et correction de copies pour le concours d'entrée en 1ère année.

2005-2006 : 192h au département de mathématiques de l'ENS Cachan en tant qu'agrégée préparatice :
- cours/TD d'analyse numérique en M1,
- cours/TD/jury d'oraux blancs/correction de copies d'analyse en préparation à l'agrégation,
- tutorat d'élève,
- secrétariat et test de sujets pour le concours d'entrée.

2003-2005 : 64h/an au département de mathématiques de l'ENS Cachan en tant que monitrice (TD et corrections de copies d'analyse en préparation à l'agrégation)

Enseignements en lycée :

2002-2003 : environ 50h de TP Maple et interrogations orales de mathématiques en classes préparatoires aux grandes écoles, Lycée Lakanal, Sceaux.

Exposés de vulgarisation :

1. mars 2009, Exposé 'Examples of control problems', lors de la visite de l'ONR au CMLA,
2. novembre 2008 : stand du CNRS lors du **forum** des entreprises de l'ENS Cachan,
3. décembre 2008 et septembre 2007 : exposé de vulgarisation d'une heure, devant les élèves en première année de mathématiques à l'ENS Cachan, sur le thème "Exemples concrets de problèmes de contrôle, résolution mathématique",
4. avril 2008 et avril 2007 : exposé de vulgarisation d'une heure, devant les élèves de terminale S et de seconde du **Lycée** du Parc Vilgénis à Massy,
5. séance de deux heures avec une classe de cinquième du **Collège** Victor Hugo à Aulnay sous Bois, dans le cadre de la Fondation 95.

Supports de cours :

Les documents suivants sont téléchargeables à l'adresse :
- *Dynamique et contrôle des systèmes nonlinéaires.* Ce document correspond au cours que j'ai donné, en collaboration avec Pierre Rouchon, au M2 MVA en 2009-2010.
- *Contrôle d'équations de Schrödinger.* Ce document correspond au cours que j'ai donné à l'école CIMPA, à Marrakech, en 2009.

- *Contrôle d'équations de Schrödinger.* Ce document correspond au cours Peccot que j'ai donné au Collège de France, en 2008.

1.4 Participation à la vie scientifique, responsabilités collectives

Responsabilités collectives :

- **membre élue du conseil scientifique de l'INSMI**, depuis juillet 2010,
- organisatrice du **séminaire** hebdomadaire du CMLA depuis septembre 2006,
- membre élue du **conseil de laboratoire** du CMLA, représentante du collège B depuis 2007.

Organisation :

- organisation d'une session, sur le thème des systèmes quantiques, au Workshop on PDEs, optimal design and numerics [Benasque, Espagne], en septembre 2005,
- organisation d'une session, sur le thème des systèmes hyperboliques, au Workshop on PDEs, optimal design and numerics [Benasque, Espagne], en septembre 2007.

Participation à des commissions de spécialistes :

- mai 2009, recrutement d'un maître de conférence au laboratoire de mathématiques de l'Université de **Versailles** Saint Quentin en Yvelines,
- juin 2006, recrutement de deux agrégés préparateurs au département de mathématiques de l'**ENS Cachan**.

Activité éditoriale :

- **éditeur associé** de MCRF (Mathematical Control and Related Fileds),
- referee pour : Mathematical Models and Methods in Applied Sciences (M3AS), ESAIM :COCV, Automatica, Journal of Optimization Theory and Applications (JOTA), Control and Cybernetics, Systems and Control Letters (SCL), Journal of Mathematical Analysis and Applications (JMAA), International Journal of Control (IJC), Annales de l'Institut Henri Poincaré : Analyse non linéaire, IEEE Transaction on Automatic control.

1.5 Séjours de recherche à l'étranger

- octobre 2006-février 2007 : post-doc sous la direction d'Enrique Zuazua à l'Universidad Autonoma de Madrid, Espagne.
- septembre-décembre 2003 : trimestre de cours à SISSA et à l'ICTP, Trieste, Italie.

1.6 Financements, réseaux

Projet MathAmsud CIP-PDE (Control and Inverse Problems in PDEs, 2009-2011) de coopération entre le Brésil (LNCC), le Chili (CONICYT) et la France (CNRS) (pour plus de détails, voir la page web : http ://docencia.mat.utfsm.cl/ ecerpa/CIP-PDE/).

GDR et GDRE CONEDP en contrôle d'EDP. Ce projet implique 30 laboratoires en France, 2 en Espagne, 1 UMI entre la France et le Chili et 26 départements en Italie. Il réunit environ 200 membres, dont 130 pour la France, 70 environ pour l'Italie ; une quarantaine sont des doctorants ou post-doctorants (pour plus de détails, voir la page web : http ://www.math.univ-metz.fr/ alabau/GDR-GDRE-CONEDP.html)

Projet ANR blanc C-QUID (Contrôle et Identification de systèmes Quantiques, 2006-2010), Ce projet est porté par une équipe de chercheurs autour de Jean-Michel Coron (Laboratoire JLL, Paris 6), Jean-Pierre Puel (Laboratoire de mathématique de l'Université de Versailles Saint Quentin), Pierre Rouchon (CAS de l'Ecole Nationale supérieure de Mines de Paris) et Gabriel Turinici (Ceremade de l'Université de Dauphine).

Projet Farman MicroNanoMAp (Micromagnétisme des Nanostructures et Mathématiques Appliquées). L'Institut Farman est un institut de l'ENS Cachan qui encourage et finance des projets pluridisciplinaires, menés conjointement par plusieurs laboratoires du campus. Les laboratoires impliqués dans le projet MicroNanoMAp sont le CMLA (Centre des Mathématiques et de Leurs Applications), avec François Alouges et Karine Beauchard, le SATIE (Systèmes et Applications des Technologies de l'Information et de l'Energie) avec Martino Lo Bue et Frederic Mazaleyrat et le LMT (Laboratoire de mécanique et technologie) avec Olivier Hubert.

Projet SIMUMATH, de la communauté autonome de Madrid : financement d'un séjour post-doctoral de 5 mois (sept 2006-jan 2007), sous la direction d'Enrique Zuazua.

1.7 Autres

- Membre de la SMAI et de la SMF.
- Langues étrangères : anglais, espagnol.

Chapitre 2

Résumé de mon activité de recherche

2.1 Introduction

2.1.1 Présentation générale

Mes travaux de recherche portent essentiellement sur l'analyse et le contrôle d'équations aux dérivées partielles (EDP). Cela signifie que j'étudie des systèmes physiques, modélisés par des EDP, sur lesquels on peut agir au moyen d'une commande ; et que je cherche des commandes permettant d'amener ces systèmes d'un état initial donné, à un état final souhaité. Afin d'introduire le vocabulaire nécessaire à cette présentation, considérons l'exemple d'une particule quantique, dans un espace de dimension n ($n \in \{1,2,3\}$), dans un potentiel $V = V(x)$, soumise à un champ électrique uniforme $u : t \in \mathbb{R} \mapsto \mathbb{R}^n$ (la commande ou le contrôle). Elle est modélisée par une fonction d'onde $\psi : \mathbb{R} \times \Omega \to \mathbb{C}$, où Ω est un domaine (possiblement non borné) de $\mathbb{R}^n$. L'évolution de cette fonction d'onde est régie par l'équation de Schrödinger

$$i\frac{\partial \psi}{\partial t}(t,x) = [-\Delta + V(x) - u(t)\mu(x)]\psi(t,x), x \in \Omega, \tag{2.1}$$

où $\mu : \Omega \to \mathbb{R}^n$ est le moment dipolaire de la particule.

La possibilité de trouver un temps $T > 0$ et un contrôle $u : [0,T] \to \mathbb{R}^n$ permettant d'amener la solution d'un état initial donné ψ_0 à un état final souhaité ψ_f est appelé le problème de contrôlabilité de l'équation de Schrödinger. On peut vouloir réaliser ce déplacement

- de façon exacte, c'est-à-dire $\psi(T) = \psi_f$ ('contrôlabilité exacte'), ou de façon approchée, c'est-à-dire $\|\psi(T) - \psi_f\| < \epsilon$, pour des paramètres ϵ arbitrairement petit et pour une certaine norme $\|.\|$ ('contrôlabilité approchée'),
- en temps fini, c'est-à-dire avec $T < +\infty$, ou asymptotiquement en temps, c'est-à-dire avec $T = +\infty$ (il s'agit alors souvent de 'stabilisation'),
- globalement, c'est-à-dire pour tous ψ_0, ψ_f vivant dans un certain espace fonctionnel ('contrôlabilité globale'), ou localement, au voisinage d'une trajectoire de référence ψ_{ref}, c'est-à-dire pour (ψ_0, ψ_f) dans un petit voisinage de $(\psi_{ref}(0), \psi_{ref}(T))$ ('contrôlabilité locale').

Une fois le problème de contrôle résolu, on s'intéresse également à d'autres propriétés qualitatives, par exemple : quel est le temps minimal pour réaliser la contrôlabilité ? y a-t-il des contrôles optimaux ? dans le cas de la stabilisation, peut-on évaluer la vitesse de convergence ? peut-on stabiliser arbitrairement vite ? etc.

Le contrôle d'EDP a de nombreuses applications ; en l'occurrence, le problème de contrôle quantique précédemment présenté est crucial en chimie quantique et pour la conception de portes logiques pour les ordinateurs quantiques.

Dans cette HDR, je me suis intéressée à des problèmes de contrôle d'EDP issus de divers domaines (mécanique quantique, équations cinétiques, micromagnétisme), et faisant intervenir diverses pathologies (contrôle bilinéaire dans les systèmes quantiques, hypoellipticité dans le cas des équations cinétiques, pathologies du flot des applications harmoniques en micromagnétisme). Je me suis également penchée sur l'utilisation d'outils, issus de la théorie du contrôle, comme, par exemple, la condition de Kalman, pour traiter des problèmes plus classiques d'analyse des EDP (comportement asymptotique pour les systèmes hyperboliques).

2.1.2 Rappel sur mon travail de thèse (A1, A2, A3) [22, 27, 28]

Ma thèse, soutenue en décembre 2005, à l'Université d'Orsay, a porté sur l'étude de la contrôlabilité et de la stabilisation de l'équation de Schrödinger, avec des commandes bilinéaires. J'ai été encadrée par Jean-Michel Coron.

Présentation de l'article (A1) [22] :

Modèle : L'article [22] porte sur l'équation de Schrödinger (2.1) avec $V = 0$, $\mu(x) = x$, $\Omega = (-1/2, 1/2)$, $n = 1$.

Résultat : J'y démontre la contrôlabilité exacte, dans un H^7-voisinage de l'état fondamental, en temps T grand, avec des contrôles dans $H_0^1(0,T)$.

Nouveauté : En raison d'un résultat négatif antérieur, dû à Ball, Marsden et Slemrod [17], de tels systèmes bilinéaires ont longtemps été considérés comme non contrôlables. Ces auteurs ont en effet démontré que, dans un cadre très général, les systèmes commandés bilinéaires ne sont pas contrôlables dans les espaces qui sont naturels pour le problème de Cauchy. Précisément, ils démontrent que, dans ces espaces, l'ensemble atteignable est d'intérieur vide (il ne peut donc coïncider avec tout l'espace, ce qui fournit la non contrôlabilité). Cependant, ce résultat négatif, bien que très intéressant, ne clôt pas la question de la contrôlabilité. Ball, Marsden et Slemrod avaient eux-même remarqué qu'on pouvait néanmoins avoir de la contrôlabilité approchée dans les espaces naturels (cad que l'ensemble atteignable pouvait être dense dans ces espaces).

Un intérêt de l'article [22] est de montrer que le résultat négatif de Ball, Marsden et Slemrod est parfois uniquement lié à un choix d'espaces fonctionnel malheureux et pas forcément à une non-contrôlabilité structurelle, comme par exemple, lorsqu'une composante évolue indépendamment du contrôle. En l'occurrence, leur résultat (après une adaptation démontrée par Turinici [160]) établit la non contrôlabilité, dans $H^2(\Omega)$, de l'équation étudiée dans [22] : l'ensemble atteignable est d'intérieur vide dans $H^2(\Omega)$. J'ai néanmoins montré que l'ensemble atteignable contient localement H^7, ce qui fournit un résultat positif de contrôlabilité dans H^7. Il est à noter que, H^7 étant d'intérieur vide dans H^2, les deux résultats cohabitent sans contradiction.

L'article [22] souligne donc l'importance du choix du cadre fonctionnel pour l'obtention de résultats positifs de contrôlabilité. Un autre intérêt de cet article est d'avoir introduit les techniques de type Nash-Moser en théorie du contrôle.

Techniques : Les techniques utilisées dans cet article sont le principe de linéarisation, la méthode du retour de Jean-Michel Coron, le théorème de Nash-Moser, la méthode des moments et les déformations quasi-statiques, que nous allons rapidement décrire ici.

Le *principe de linéarisation* est l'approche classique pour démontrer la contrôlabilité locale d'un système non linéaire au voisinage d'une trajectoire de référence :

- on démontre d'abord la contrôlabilité du système linéarisé autour de cette trajectoire,
- on applique ensuite un théorème d'inversion locale (ou de point fixe) pour conclure.

La *méthode des moments* intervient dans la première partie de cette stratégie. elle consiste à reformuler le problème de contrôlabilité pour un système linéaire en un problème de moments sur le contrôle. En l'occurrence, dans [22], il s'agit d'un problème de moment trigonométriques, dont les exponentielles vérifient une inégalité de Ingham convenable (voir [99]).

Parfois, le système linéarisé n'est pas contrôlable, on peut alors essayer d'appliquer la *méthode du retour de Jean-Michel Coron*. Elle consiste à

- trouver une autre trajectoire de référence, bénéficiant de meilleures propriétés de contrôlabilité (typiquement, admettant un linéarisé contrôlable),
- justifier qu'on peut se déplacer de la première trajectoire de référence à la deuxième et inversement ; pour cela, dans [22], on utilise des déformations quasi-statiques.

Parfois, les résultats dont on dispose concernant la contrôlabilité du système linéarisé et le caractère bien posé du système non linéaire ne sont pas suffisants pour conclure avec le théorème d'inversion locale. Par exemple, dans [22], en exploitant les outils classiques, le problème de Cauchy ne semble pas bien posé dans les espaces adaptés à la contrôlabilité du linéarisé : il y a une apparente perte de régularité. C'est la raison pour laquelle nous avons appliqué le *théorème de Nash-Moser*. Ce théorème nécessite essentiellement 3 hypothèses :

- des familles d'espaces $(E_a)_{a \geqslant 0}$, $(F_b)_{b \geqslant 0}$, munis d'opérateurs régularisants convenables,
- une application non linéaire $\Theta : E_a \rightarrow F_a$ de classe C^2 avec une borne appropriée sur $d^2\Theta$,
- une différentielle $d\Theta(x)$ surjective pour tout x dans un petit voisinage de 0, avec une 'estimée douce' sur $d\Theta(x)^{-1}$.

La troisième hypothèse est souvent la plus difficile et la plus technique à vérifier. Dans notre cas, il s'agit de démontrer la contrôlabilité d'une infinité de systèmes linéaires, avec une borne convenable sur les contrôles utilisés.

Présentation de l'article (A2) [27] :

Modèle : L'article [27], co-écrit avec Jean-Michel Coron, porte sur l'équation de Schrödinger (2.1) avec $V = 0$, $\mu(x) = x$, $\Omega = (-1/2, 1/2)$, $n = 1$, couplée avec l'équation différentielle

$$\left\{ \begin{array}{l} \frac{dS}{dt}(t) = u(t), \\ \frac{dD}{dt}(t) = S(t). \end{array} \right. \tag{2.2}$$

Ce système couplé modélise une particule quantique dans un puits de potentiel carré et infini (la 'boite'), en translation. La position de la boite est repérée par la variable $D(t)$, sa vitesse par la variable $S(t)$. Le contrôle $u(t)$ correspond donc à l'accélération de la boite, c'est-à-dire à la force qu'on lui applique (voir [148] pour la modélisation).

Résultat : Nous démontrons la contrôlabilité exacte entre états propres de ce système couplé : étant donnés deux états propres pour la particule ψ_0 et ψ_f, et une position D_f pour la boite, il existe un temps $T > 0$ et un contrôle $u : [0, T] \to \mathbb{R}$ permettant d'amener simultanément la particule de l'état initial ψ_0, à l'état final ψ_f et la boite de la configuration initiale $(D, S)(0) = (0, 0)$ à la configuration finale $(D, S)(T) = (D_f, 0)$.

Nouveauté : La nouveauté essentielle de ce résultat réside dans le caractère global du résultat obtenu, dans le cadre de la contrôlabilité exacte d'un système bilinéaire de dimension infinie.

Techniques : La preuve repose sur des résultats de contrôlabilité locale, couplés avec un argument de compacité. Pour démontrer les résultats de contrôlabilité locale, en plus de techniques utilisées dans l'article [22], nous avons utilisé des développements à l'ordre deux. En effet, en raison de l'introduction des variables S et D, certaines directions ne sont plus contrôlables sur les systèmes linéarisés considérés et nous les récupérons en utilisant des termes d'ordre supérieur (l'ordre deux suffit ici, mais l'ordre 3 est parfois nécessaire comme, par exemple dans [66]).

Présentation de l'article (A3) [28] :

Modèle : Dans [28], Jean-Michel Coron, Mazyar Mirrahimi, Pierre Rouchon et moi considérons une équation de Schrödinger de dimension finie, c'est-à-dire l'équation différentielle correspondant aux approximants de Galerkin de l'équation (2.1).

Résultat/Nouveauté : Nous étudions la stabilisation de l'état fondamental, à l'aide de lois feedback. Dans un article antérieur [134], Mazyar Mirrahimi, Pierre Rouchon et Gabriel Turinici proposent des loi feedback explicites qui stabilisent l'état fondamental, lorsque le système linéarisé autour de cet état est contrôlable. Dans l'article [28], nous nous penchons sur un cas dégénéré, où le linéarisé autour de l'état fondamental n'est pas contrôlable. Nous proposons des lois feedback implicites qui stabilisent l'état fondamental, sous des hypothèses convenables.

Techniques : Les fonctions de Lyapunov sont un outil classique pour étudier la stabilité d'un équilibre d'un système dynamique. Dans le cas des systèmes commandés, le contrôle est à notre disposition donc il y a plus de 'chance' qu'une fonction donnée puisse être une fonction de Lyapunov, pour un choix convenable de loi feedback. En conséquence, les fonctions de Lyapunov sont énormément utilisées pour la stabilisation de systèmes commandés et c'est l'outil exploité ici. La preuve de la convergence repose sur le principe d'invariance de LaSalle.

2.1.3 Structure

Ce dossier préliminaire présente mes travaux récents.

Les Sections 2.2, 2.3 et 2.4 s'inscrivent dans le prolongement thématique de ma thèse tout en introduisant des outils nouveaux. Plus précisément, la Section 2.2 présente des résultats de contrôlabilité exacte pour des systèmes bilinéaires de dimension infinie dont le spectre est discret. La Section 2.3 est consacrée à des résultats de stabilisation pour des équations de Schrödinger bilinéaires, et la Section 2.4 porte sur la contrôlabilité exacte du système linéarisé de (2.1) autour d'un état propre, en 2D et 3D.

Les Sections 2.5, 2.6, 2.7 et 2.8 portent sur des modèles ou des sujets nouveaux. La Section 2.5 est consacrée à la contrôlabilité et la stabilisation de l'équation de Bloch, qui est un autre exemple de système quantique. La Section 2.6 traite de systèmes hyperboliques partiellement dissipatifs. Il s'agit ici de traiter des problèmes classiques de la théorie des EDP (comportement asymptotique, existence de solutions globales régulières) grâce à des outils issus de la théorie du contrôle. Dans la Section 2.7, on présente un résultat de contrôlabilité pour une équation cinétique linéaire hypocoercive, l'équation de Kolmogorov. Enfin, la Section 2.8 porte sur l'analyse et le contrôle d'une EDP issue du micromagnétisme, l'équation de Landau-Lifschitz.

2.2 Contrôlabilité exacte de systèmes bilinéaires (A4, A5, A12, A13) [23, 24, 31, 25]

Après ma thèse, dans un premier temps, je me suis attachée à développer davantage les techniques acquises au cours de ma thèse :

- dans [23], je les ai appliquées à une autre équation de Schrödinger, de type bilinéaire, mais dans laquelle l'interaction entre le contrôle et la fonction d'onde se fait de façon un peu plus non linéaire,
- dans [24], je les ai utilisées sur une équation des poutres 1D, qui est l'un des exemples historiques auxquels le résultat négatif de Ball, Marsden et Slemrod s'appliquait.

Dans un deuxième temps, je me suis concentrée sur la simplification de ces preuves. En effet, toutes reposaient sur l'utilisation du théorème de Nash-Moser, pour gérer une apparente perte de régularité. La technicité de son application limitait l'adaptation de cette stratégie à des situations plus compliquées (par exemple multi-D, ou impliquant des non linéarités). De plus, beaucoup de problèmes mathématiques, initialement résolus avec cette méthode se sont avérés ensuite résolvable avec le théorème d'inversion locale classique ; l'exemple le plus célèbre étant le problème de plongement isométrique (toute variété Riemannienne peut être isométriquement plongée dans un espace $\mathbb{R}^n$). John Nash a introduit les méthodes de Nash-Moser pour démontrer ce résultat en 1956 [136]. Cependant Matthias Günther a trouvé en 1990 une façon de démontrer ce résultat à partir du théorème d'inversion locale classique [94, 95]. Nous sommes finalement ici dans une situation similaire : le théorème de Nash-Moser nous a fait 'gagner du temps', mais n'est pas nécessaire. Les articles [25] et [31] portent sur cette simplification ; dans chaque cas, un effet régularisant permet de conclure avec le théorème d'inversion locale :

- d'abord, dans [25], j'étudie la contrôlabilité exacte d'une équation des ondes linéaire, avec contrôle bilinéaire,
- ensuite, dans [31], Camille Laurent et moi présentons une preuve simplifiée d'une version optimale de mon résultat de thèse [22], ainsi que des généralisations à des équations non linéaires.

Présentation de l'article (A4) [23] :

Modèle : Dans cet article, je considère une particule quantique dans une puits de potentiel carré infini, de longueur variable. Il s'agit d'un système commandé non linéaire dans lequel l'état est la fonction d'onde $\psi(t,x)$ de la particule et le contrôle est la longueur $l(t)$ du puits.

Résultat : Je démontre le résultat de contrôlabilité exacte suivant : étant donné un état initial ψ_0 assez proche dans H^{5+} d'un état propre, étant donné un état final ψ_f assez proche dans H^{5+} d'un (éventuellement autre) état propre, il existe un temps $T>0$ et une fonction continue $l:[0,T]\to(0,+\infty)$ telle que $l(0)=l(T)=1$, qui amène la fonction d'onde de ψ_0 à ψ_f, en temps T. En particulier, on peut faire passer la particule d'un niveau d'énergie à un autre en agissant sur la longueur du puits de façon convenable.

Nouveauté : Un intérêt de ce travail consiste à ramener un problème fortement non linéaire à un problème de contrôle de type bilinéaire, ce qui permet d'exploiter les techniques précédemment décrites. L'interaction entre l'état et le contrôle est cependant plus non linéaire que dans les précédents travaux, ce qui nécessite quelques adaptations. Enfin, un gain sur la régularité est fait par rapport aux précédents travaux : avec des contrôles H^1_0, la contrôlabilité est ici démontrée dans H^{5+}, alors qu'elle était démontrée seulement dans H^7 dans [22] et [27] ; ceci est permis par une application plus fine du théorème de Nash-Moser.

Techniques : Dans un premier temps, on transforme, grâce à des changements de variable adaptés, l'équation posée sur le domaine variable $(0,l(t))$ en une équation posée sur le domaine fixe $(0,1)$. Des changements de fonction d'onde et de contrôle permettent alors d'aboutir à une équation de type bilinéaire, mais dans laquelle l'interaction entre l'état et le contrôle est plus non linéaire que dans les précédents travaux (en $(\dot{u}-4u^2)(t)\mu(x)\psi(t,x)$ au lieu de $u(t)\mu(x)\psi(t,x)$).

Comme dans [27], la stratégie globale repose sur un argument de compacité, qui nécessite la contrôlabilité locale au voisinage de nombreuses trajectoires périodiques. Ces résultats locaux sont démontrés par linéarisation, grâce au théorème de Nash-Moser. Pour certaines trajectoires, le système linéarisé n'est pas complètement contrôlable et perd certaines directions. On utilise alors des termes d'ordre deux pour les récupérer.

Présentation de l'article (A5) [24] :

Modèle : Dans cet article, je considère une poutre homogène, horizontale, encastrée aux deux extrémités, soumise à une charge axiale $p(t)$. Sa déformation verticale au point $x\in(0,1)$ est repérée par la quantité $u(t,x)$, qui résout l'équation des poutres

$$\begin{cases} u_{tt}(t,x)+u_{xxxx}(t,x)+p(t)u_{xx}(t,x)=0, x\in(0,1), \\ u(t,0)=u(t,1)=u_x(t,0)=u_x(t,1)=0. \end{cases}$$

Résultat : Je démontre la contrôlabilité exacte locale, dans un $H^{5+}\times H^{3+}$-voisinage d'une trajectoire du système libre, avec des contrôles $p\in H^1_0(0,T)$ et en temps $T=8/\pi$.

Nouveauté : La nouveauté de ce travail réside essentiellement dans l'adaptation des techniques développées pour l'équation de Schrödinger, à une autre équation.

Techniques : La preuve repose sur le principe de linéarisation et le théorème de Nash-Moser.

Présentation de l'article (A13) [25] :

Modèle : Dans cet article, je considère une équation des ondes avec contrôle bilinéaire

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) + u(t)\mu(x)w(t,x), x \in (0,1), \\ w_x(t,0) = w_x(t,1) = 0, \end{cases} \tag{2.3}$$

dans laquelle l'état est le couple (w, w_t) et le contrôle est $u : [0,T] \to \mathbb{R}$. Il s'agit d'un modèle de corde vibrante, soumise à une charge axiale $u(t)\mu(x)$.

Résultats : J'étudie la contrôlabilité exacte locale, dans un $H^3 \times H^2$-voisinage de la trajectoire de référence constante en 1, avec des contrôles $L^2(0,T)$. Précisément, je montre que, sous des hypothèses génériques sur μ,

- pour $T > 2$, cette contrôlabilité a lieu,
- pour $T < 2$, cette contrôlabilité n'a pas lieu : l'ensemble atteignable est alors localement contenu dans une sous-variété non plate de $H^3 \times H^2$, de codimension infinie.

Lorsque $T = 2$, le système est contrôlable 'à codimension un près' : on peut contrôler le couple $(w - \int_0^1 w, w_t)$, mais on ne peut alors pas contrôler la moyenne de w.

Nouveauté : La première nouveauté de ce travail est de démontrer de la contrôlabilité exacte pour un système bilinéaire de dimension infinie, sans appliquer le théorème de Nash-Moser. En effet, un effet régularisant est facile à mettre en évidence et il permet de conclure avec le théorème d'inversion locale.

La deuxième nouveauté de cet article réside dans l'obtention d'un résultat négatif fort : le résultat de non contrôlabilité pour $T < 2$ est plus fort que les résultats négatifs démontrés par Ball, Marsden et Slemrod [17]. En effet, il exclut que l'ensemble atteignable puisse coïncider (localement) avec un espace fonctionnel strictement plus régulier. Ce type de résultat négatif avait été préalablement démontré pour l'équation de Bloch dans l'article [29], qui sera présenté dans la Section 2.5.

Un autre intérêt de ce résultat est de faire progresser la compréhension de la contrôlabilité de l'équation de Schrödinger 2D. En effet, l'équation (2.3) peut être vue comme un toy model, du point de vue spectral, pour l'équation (2.1) avec $V = 0$ et $\Omega \subset \mathbb{R}^2$. La répartition des valeurs propres du Laplacien sur un ouvert 2D générique étant assez mal connue, la contrôlabilité exacte de cette équation de Schrödinger est un problème ambitieux. Pour (2.3), le spectre vérifie la même formule de Weyl, mais présente plus de structure, ce qui rend l'analyse plus accessible.

Techniques : La preuve des trois résultats repose sur le principe de linéarisation. Lorsque $T > 2$, on montre que le linéarisé autour de la trajectoire de référence est contrôlable et on conclut avec le théorème d'inversion locale. Lorsque $T < 2$, on montre que l'ensemble atteignable pour le système linéarisé est un sous-espace vectoriel de $H^3 \times H^2$ de codimension infinie. Le théorème des fonctions implicites justifie alors que l'ensemble atteignable, pour le système non linéaire, est une sous-variété de $H^3 \times H^2$ de codimension infinie. On montre ensuite qu'elle ne coïncide pas avec son espace tangent en $(1,0)$ (sous-variété 'non plate') en utilisant les termes d'ordre deux. Lorsque $T = 2$, on montre que le système linéarisé est contrôlable à codimension un près. Le théorème d'inversion locale permet alors de montrer que l'ensemble atteignable, pour le système non linéaire, est alors une sous-variété de $H^3 \times H^2$ de codimension un.

L'effet régularisant se démontre avec des techniques élémentaires : intégrations par parties, égalité de Bessel Parseval.

Présentation de l'article (A12) [31] :

Modèles : Cet article, co-écrit avec Camille Laurent, porte sur l'équation de Schrödinger (2.1) avec $V = 0$, $\Omega = (0, 1)$, $n = 1$. On y étudie également des équations de Schrödinger nonlinéaires

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} \pm |\psi|^2\psi - u(t)\mu(x)\psi, x \in (0, 1) \tag{2.4}$$

et des équations des ondes nonlinéaires

$$w_{tt} = w_{xx} + f(w, w_t) + u(t)\mu(x)(w + w_t), x \in (0, 1). \tag{2.5}$$

Résultat : Dans un premier temps, on démontre la contrôlabilité exacte locale, dans un H^3-voisinage de l'état fondamental, de l'équation de Schrödinger linéaire, avec des contrôles dans $L^2(0, T)$ et en tout temps $T > 0$. On montre ensuite un résultat analogue dans des espaces plus réguliers (contrôlabilité dans H^5 avec des contrôles H^1_0) et pour une équation de Schrödinger linéaire posée sur la boule unité de $\mathbb{R}^3$, avec des données radiales.

Dans un deuxième temps, on démontre des résultats analogues, mais pour les équations non linéaires (2.4) et (2.5).

Nouveauté : La première nouveauté de ce travail réside dans l'obtention de contrôlabilité exacte pour des équations de Schrödinger bilinéaires, sans le théorème de Nash-Moser (la situation est ici un peu moins favorable que dans le précédent article [25]).

De plus, le résultat obtenu est optimal du point de vue des espaces fonctionnels et du temps de contrôlabilité (arbitrairement petit, dès que le linéarisé est contrôlable) : ceci constitue une autre amélioration par rapport à [22].

Enfin, cette preuve est assez robuste pour gérer d'éventuelles non linéarités et nous l'appliquons à des équations de Schrödinger non linéaires et des équations des ondes non linéaires.

Techniques : Nous mettons en évidence un effet régularisant caché, montrant qu'il n'y a donc pas perte de régularité et que le théorème d'inversion locale classique suffit pour conclure. Cet effet régularisant se démontre avec des outils élémentaires : intégrations par parties et inégalités de Ingham.

2.3 Stabilisation d'équations de Schrödinger (A6, A15) [32, 33]

Les articles **(A6, A15)** [32, 33] portent sur la stabilisation asymptotique de l'état fondamental, pour des EDP de Schrödinger bilinéaires, avec des contrôles en boucle fermée (lois feedback). Ce problème a été traité lorsque l'EDP est remplacée par une EDO [134, 28] ;

dans ce cas, la preuve de la convergence repose sur le principe d'invariance de LaSalle. Ce principe est un outil puissant pour étudier la stabilité asymptotique d'un équilibre d'une EDO, mais son utilisation en dimension infinie est plus délicate (parce que les fermés bornés ne sont pas nécessairement compacts). Si on veut obtenir, en dimension infinie, le résultat de stabilisation forte analogue à celui connu en dimension finie, alors on a besoin d'une propriété de compacité des trajectoires [67], ou bien d'une fonction de Lyapunov stricte [68], qui peuvent être difficile à établir. Dans cette section, nous proposons deux autres adaptations du principe de LaSalle à la dimension infinie : elles évitent de démontrer la propriété de compacité, mais, en contrepartie, elles ne permettent d'obtenir qu'un résultat plus faible de stabilisation : stabilisation approchée (A6) [32] ou stabilisation faible (A15) [33], comme dans [18].

Présentation de l'article (A6) [32] :

Modèle : Mazyar Mirrahimi et moi considérons l'équation (2.1) avec $V(x) = \gamma x$, où γ est une petite constante réelle, $\Omega = (-1/2, 1/2)$, $\mu(x) = x$.

Résultat : Nous proposons des lois feedback explicites qui réalisent la stabilisation approchée (dans L^2) et semi-globale de l'état fondamental.

Nouveauté : L'intérêt essentiel de ce résultat, par rapport aux précédents articles de la littérature, est qu'on travaille sur une EDP de Schrödinger (et pas sur une équation différentielle). De plus, notre résultat est global et les contrôles sont explicites.

Techniques : La synthèse des loi feedback repose sur l'utilisation de fonctions de Lyapunov. Elle est proche de celle utilisée dans [131].

La situation la plus simple correspond au cas où le paramètre γ est non nul. Alors la fonction de Lyapunov utilisée est inspirée de la distance L^2 à la cible, mais on lui ajoute un terme supplémentaire, faisant intervenir les N premières composantes de la fonction d'onde, et dont le rôle est double : 1. éviter la perte de masse dans les hautes fréquences, 2. favoriser la croissance de la population dans l'état fondamental. L'indice de coupure N dépend alors de la condition initiale ; le choix de ce paramètre permet d'obtenir la stabilisation semi-globale. Les lois feedback obtenues sont complètement explicites.

La situation où le paramètre γ est nul est plus compliquée, du fait de la non contrôlabilité du système linéarisé autour de l'état fondamental (qui empêche que l'ensemble invariant soit réduit à la cible). Dans ce cas dégénéré, en combinant convenablement les idées du précédent paragraphe (pour $\gamma \neq 0$) et les idées développées en dimension finie dans [28], nous proposons des lois feedback, définies implicitement, qui stabilisent approximativement l'état fondamental.

Présentation de l'article (A15) [33] :

Modèle : Vahagn Nersesyan et moi considérons l'équation (2.1) où Ω est un ouvert borné régulier de $\mathbb{R}^n$ et $V, \mu \in C^\infty(\overline{\Omega}, \mathbb{R})$.

Résultat/Nouveauté : Récemment, Nersesyan a proposé des lois de rétroaction explicites et démontré l'existence d'une suite de temps $(t_n)_{n\in\mathbb{N}}$ auxquels les valeurs de la solution du système bouclé convergent faiblement dans H^2 vers l'état fondamental. Ici, nous démontrons la convergence faible dans H^2 de toute la solution, quand $t \to +\infty$, ce qui fournit la stabilisation faible et semi-globale de l'état fondamental.

Techniques : Comme dans [138], la synthèse des lois feedback repose sur l'utilisation de fonctions de Lyapunov et l'ensemble invariant de LaSalle est bien réduit à la cible. Dans cet article, nous démontrons que les seules valeurs d'adhérence faible H^2 sont sur l'état fondamental, ce qui implique la convergence de tout le profil. Le point essentiel est que les lois feedback sont bien définies pour des fonctions strictement moins régulières que H^2 (formellement $H^{3/2}$ suffit).

Remarque : D'autres résultats de contrôlabilité approchée, pour des modèles similaires ont été démontrés, par Chambrion, Mason, Sigalotti et Boscain [54] avec des méthodes de contrôle géométrique et par Ervedoza et Puel [80] avec un modèle réduit.

2.4 Généricité et contrôle spectral pour une équation de Schrödinger linéaire (A9) [26]

Modèle : Dans cet article, Yacine Chitour, Djalil Khateb, Ruixing Long et moi étudions le système linéarisé, autour de l'état fondamental, de l'équation (2.1) avec $V = 0$, $\Omega \subset \mathbb{R}^2$ ou $\mathbb{R}^3$, c'est-à-dire l'équation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\Delta\psi(t,x) - u(t)\mu(x)\varphi_1(x)e^{-i\lambda_1 t}, x \in \Omega, \\ \psi(t,x) = 0, x \in \partial\Omega, \end{cases}$$

où λ_1 est la première valeur propre du Laplacien sur Ω, φ_1 est le vecteur propre associé et $\mu : \Omega \to \mathbb{R}$.

Motivation : En 2D, la contrôlabilité exacte de ce système est une question difficile. En effet, elle se reformule sous forme d'un problème de moments trigonométriques sur le contrôle, où les fréquences sont les valeurs propres du Laplacien. Une condition nécessaire et suffisante à l'existence d'une solution dans $L^2(0,T)$, pour un tel problème de moment, est l'existence d'un gap uniforme (dans un sens faible : voir [164]) entre les fréquences. Or, l'existence d'un tel gap, pour les valeurs propres du Laplacien sur un domaine 2D générique, est un problème ouvert.

La contrôlabilité exacte étant un problème difficile, il est naturel de se pencher sur des notions de contrôlabilité plus faibles. En l'occurrence, nous étudions la contrôlabilité spectrale, en temps fini T (c'est-à-dire la contrôlabilité entre sommes finies de vecteurs propres du Laplacien).

Résultat/Techniques : Des outils élémentaires d'analyse permettent de déterminer une condition nécessaire, appelée (Kal) (parce qu'elle est de type Kalman) pour la contrôlabilité spectrale. Sur un ouvert 2D vérifiant la condition (Kal), on démontre qu'il existe un temps $T_{min}(\Omega) > 0$ (dépendant de la densité des valeurs propres du Laplacien sur Ω) en dessous duquel la contrôlabilité spectrale n'a pas lieu et en dessus duquel elle a lieu. Sur un ouvert 3D général, on montre que la contrôlabilité spectrale n'a pas lieu (quel que soit la valeur de $T > 0$). Ces résultats reposent sur un travail antérieur de Haraux et Jaffard [100]. On démontre également un résultat négatif pour le système 3D, couplé avec l'équation différentielle (2.2) ; la preuve repose ici sur de l'analyse complexe, comme dans [57].

Ensuite, on s'intéresse à la généricité, par rapport au domaine $\Omega \subset \mathbb{R}^2$ de la condition (Kal). Le Lemme de Baire permet une première réduction du problème. Mais les techniques standard de dérivation par rapport au domaine ne sont pas suffisantes pour conclure. Nous utilisons alors une étude fine des opérateurs Neumann-Dirichlet associés à certaines équations de Helmholtz.

2.5 Contrôle et stabilisation de l'équation de Bloch (A10, A14) [29, 30]

Modèle : L'équation de Bloch modélise un ensemble de spins, dans un champ magnétique

$$B(t) = (u(t), v(t), B_0),$$

sans interaction entre eux et présentant une dispersion dans leur fréquence de Larmor $\omega \in (\omega_*, \omega^*)$. Chaque spin est représenté par un vecteur $M = M(t,\omega) \in \mathbb{S}^2$ dont l'évolution est régie par l'équation de Bloch

$$\frac{\partial M}{\partial t}(t,\omega) = [u(t)e_1 + v(t)e_2 + \omega e_3] \wedge M(t,\omega), \omega \in (\omega_*, \omega^*), \tag{2.6}$$

où $-\infty < \omega_* < \omega^* < +\infty$, (e_1, e_2, e_3) est la base canonique de $\mathbb{R}^3$, $\wedge$ désigne le produit vectoriel de $\mathbb{R}^3$. C'est un système commandé bilinéaire, de dimension infinie, dans lequel
- l'état est la fonction M,
- le contrôle est le couple $(u, v) : [0, T] \to \mathbb{R}^2$.

Il s'agit donc d'étudier la contrôlabilité simultanée d'un continuum d'équations différentielles, paramétrées par $\omega \in (\omega_*, \omega^*)$. Ce problème a été introduit par Li et Khaneja dans [125].

Intérêt : L'équation de Bloch présente, en outre, l'intérêt d'être un prototype pour les systèmes bilinéaires de dimension infinie admettant du spectre continu. En effet, le spectre de l'opérateur $\mathcal{A}$, défini par

$$(\mathcal{A}M)(\omega) := \omega e_3 \wedge M(\omega),$$

est formellement $-i(\omega_*, \omega^*) \cup i(\omega_*, \omega^*)$: pour tout $\omega_\sharp \in (\omega_*, \omega^*)$, le vecteur propre associé à $\pm i\omega_\sharp$ est $(1, \mp i, 0)^t \delta_{\omega_\sharp}(\omega)$. L'étude de l'équation de Bloch est donc également motivée par la compréhension de la contrôlabilité des équations de Schrödinger (2.1) admettant du spectre continu.

Présentation de l'article (A10) [29] :

Résultat/Techniques : Dans cet article, Jean-Michel Coron, Pierre Rouchon et moi présentons plusieurs résultats concernant le contrôle de l'équation de Bloch, en distinguant bien la contrôlabilité exacte, de la contrôlabilité approchée ; le contrôlabilité en temps fini, de la contrôlabilité en temps infini ; la contrôlabilité avec des contrôles bornés a priori (dans L^2)

de la contrôlabilité avec des contrôles non bornés a priori.

Dans un premier temps, nous étudions la contrôlabilité exacte, en temps fini T, avec des contrôles bornés a priori dans $L^2(0,T)$, au voisinage de la trajectoire de référence constante en e_3.

Lorsque l'équation est posée sur tout l'espace (c'est-à-dire lorsque $\omega_* = -\infty$ et $\omega^* = +\infty$), il n'y a pas contrôlabilité exacte : l'ensemble atteignable est une sous-variété non plate de l'espace fonctionnel $C_b^0 \cap L^2(\mathbb{R})$. Ce résultat de non contrôlabilité est très fort : il exclut, par exemple, que l'ensemble atteignable puisse coïncider avec un espace fonctionnel strictement plus régulier que $C_b^0 \cap L^2(\mathbb{R})$. La preuve repose sur le théorème d'inversion locale. L'argument essentiel est que la différentielle de la end-point map en zéro est injective et non surjective, parce qu'elle coïncide avec la transformée de Fourier.

Lorsque l'équation est posée sur un intervalle borné (comme il se doit), un argument d'analyticité par rapport à la variable ω permet de montrer qu'il existe des cibles analytiques arbitrairement petites qui ne peuvent être atteintes. Là encore, il s'agit d'un résultat de non contrôlabilité très fort, puisqu'il est indépendant de la régularité.

L'équation de Bloch n'étant pas contrôlable, de façon exacte, avec des contrôles bornés a priori dans L^2, dans un deuxième temps, nous étudions sa contrôlabilité avec des contrôles non bornés a priori, typiquement des sommes de masses de Dirac.

Nous démontrons que l'équation de Bloch est globalement contrôlable de façon approchée, dans $H^1(\omega_*, \omega^*)$, en temps fini, avec de tels contrôles. La preuve repose sur un argument de non commutativité, démontré par Li et Khaneja dans [125] et un argument variationnel. Ce résultat présente l'intérêt d'être global et de garantir la contrôlabilité approchée uniformément par rapport à $\omega \in (\omega_*, \omega^*)$, mais il présente l'inconvénient de ne pas être constructif.

En conséquence, nous proposons ensuite des contrôles (non bornés) explicites, qui réalisent la contrôlabilité exacte à e_3, en temps infini, localement au voisinage de e_3 (c'est-à-dire pour des conditions initiales suffisamment proches de e_3). La preuve repose sur la méthode du retour de Jean-Michel Coron et l'analyse de Fourier.

Nouveauté : La nouveauté essentielle de ce travail réside dans la gestion d'un spectre continu, pour l'obtention de résultats de contrôlabilité, sur un système bilinéaire de dimension infinie. Ce travail semble indiquer que la présence d'un spectre continu empêche la contrôlabilité exacte, avec des contrôles bornés a priori (dans L^2) ; cette contrôlabilité pouvant être récupérée avec des contrôles non bornés a priori.

Présentation de l'article (A14) [30] :

Résultat : Dans cet article, Paolo Sergio Pereira da Silva, Pierre Rouchon et moi étudions la stabilisation de l'équation de Bloch. Nous proposons des lois feedback explicites qui stabilisent localement le système sur l'état de spin uniforme $-1/2$ (c'est-à-dire $M \equiv -e_3$).

Techniques : La synthèse des lois feedback repose sur l'utilisation de fonctions de Lyapunov de type H^1. Les contrôles sont la superposition d'une séquence de masses de Dirac munie d'une structure périodique en temps et d'une loi feedback régulière. La séquence de masses de Dirac est choisie astucieusement pour réduire la dispersion naturelle du système.

La convergence a lieu au sens de la topologie H^1 faible et pour des conditions initiales dans un petit voisinage H^1 de e_3. La preuve de cette convergence repose sur une adaptation du principe d'invariance de LaSalle en dimension infinie.

Le principe d'invariance de LaSalle est un outil puissant pour montrer la stabilité asymptotique d'un équilibre d'un système dynamique de dimension finie : essentiellement, il suffit de vérifier que l'ensemble invariant est réduit au point d'équilibre. Pour un système de dimension infinie (une EDP ou un continuum d'équations différentielles, en l'occurrence), l'utilisation du principe de LaSalle est plus délicate, parce que les fermés bornés ne sont plus compacts. Deux stratégies sont alors possibles pour cette adaptation :

- soit on se satisfait d'un résultat plus faible (par exemple de la stabilisation approchée, comme dans [32], ou de la stabilisation faible, comme dans [18]) et dans ce cas montrer que l'ensemble invariant est réduit à la cible peut-être suffisant,
- soit on veut la stabilisation forte, et dans ce cas, il est nécessaire de démontrer une propriété supplémentaire de compacité sur les trajectoires du système bouclé (c'est ce qui est fait dans [67]) ; une autre façon de procéder consiste à produire une fonction de Lyapunov stricte (comme, par exemple, dans [68]).

Le résultat de **(A14)** [30], se situe donc dans la première catégorie. Il est à noter que le résultat de convergence de [18] ne s'applique pas à notre cas : la preuve de la convergence utilise donc des arguments nouveaux.

2.6 Systèmes hyperboliques (A11) [35]

<u>Modèle :</u> Dans cet article, Enrique Zuazua et moi considérons des systèmes hyperboliques partiellement dissipatifs. Dans un premier temps, nous étudions des systèmes linéaires à coefficients constants partiellement dissipatifs

$$\frac{\partial w}{\partial t}(t,x) + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j}(t,x) = -Bw(t,x) \tag{2.7}$$

où $A_1, ..., A_m$ sont des matrices réelles $n \times n$ symétriques, et B est une matrice $n \times n$ de la forme

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, D \in \mathbb{R}^{n_2 \times n_2}, X^t D X > 0, \forall X \in \mathbb{R}^{n_2} - \{0\},$$

où le premier bloc diagonal de B est de taille $n_1 \times n_1$ et $n_1 + n_2 = n$. Dans un deuxième temps, nous étudions des systèmes de loi de conservation à n composantes, dans un espace de dimension m de la forme

$$\frac{\partial w}{\partial t}(t,x) + \sum_{j=1}^{m} \frac{\partial F_j(w)}{\partial x_j}(t,x) = Q(w)(t,x) \tag{2.8}$$

où $Q, F_1, ..., F_m : \mathbb{R}^n \to \mathbb{R}^n$ sont des fonctions régulières et

$$Q(w) = \begin{pmatrix} 0 \\ q(w) \end{pmatrix}, q(w) \in \mathbb{R}^{n_2}.$$

<u>Résultats et techniques pour les systèmes linéaires (2.7) :</u> Dans une première partie, nous analysons les comportements asymptotiques possibles pour (2.7). Les solutions de ce système

s'expriment explicitement, grâce à la transformée de Fourier

$$\hat{w}(t,\xi) = \exp[E(\xi)t]\hat{w}_0(\xi), \quad E(\xi) := -B - iA(\xi), \quad A(\xi) := \sum_{j=1}^{m} \xi_j A_j.$$

Lorsque $n_2 \neq n$, la matrice $E(\xi)$ n'est pas coercive. Cependant, il est bien connu que l'interaction entre le terme de dissipation $-Bw$ et la dynamique du système peut éventuellement dissiper toutes les composantes. En l'occurrence, Shizuta et Kawashima [151] ont démontré que, sous la condition

$$\text{(SK)} : \forall \xi \in \mathbb{R}^m, \text{Ker}(B) \cap \{ \text{ vecteurs propres de } A(\xi)\} = \{0\},$$

on a

$$\exists C, c > 0 \text{ tels que } \exp[E(\xi)t] \leqslant Ce^{c\min\{1,|\xi|^2\}t}, \forall \xi \in \mathbb{R}^m, \forall t \in [0,+\infty). \tag{2.9}$$

Ceci permet de montrer que toute solution de (2.7) associée à une condition initiale $w^0 \in L^1 \cap L^2(\mathbb{R}^m)$ se décompose en

$$w = w_1 + w_2 \text{ où } \|w_1(t)\|_{L^2} \leqslant Ce^{-\lambda t}\|w^0\|_{L^2} \text{ et } \|w_2\|_{L^\infty} \leqslant Ct^{-\frac{m}{2}}\|w^0\|_{L^1}, \forall t \in [0,+\infty),$$

w_1 comportant les hautes fréquences et w_2 les basses fréquences de w_0. La condition (SK) est donc une condition suffisante pour que le terme dissipatif affecte toutes les composantes du système.

Dans l'article [35], nous proposons une preuve plus simple du résultat (2.9) de Shizuta et Kawashima, qui permet également d'étendre l'analyse aux situations où la condition (SK) n'est pas vérifiée. Pour cela, nous construisons une fonction de Lyapunov explicite pour l'équation différentielle (à ξ fixé)

$$\frac{dx}{dt}(t) = E(\xi)x(t).$$

La construction de cette fonction de Lyapunov est similaire à celles introduites par Villani dans [166] et tire profit de la condition de Kalman de la théorie du contrôle. Elle permet d'obtenir une inégalité du type

$$\|x(t)\| \leqslant 2\|x_0\|e^{-c\min\{1,\rho^2\}N(\omega)t}, \forall t \in [0,+\infty). \tag{2.10}$$

où $\xi = \rho\omega$, $\rho > 0$, $\omega \in \mathbb{S}^{m-1}$, $c > 0$ est indépendant de ξ et $N : \mathbb{S}^{m-1} \to [0,+\infty)$ est une fonction explicite (faisant intervenir les matrices $A_1, ..., A_m, B$). L'équivalence entre la condition (SK) et la condition de Kalman permet de montrer que la fonction N est uniformément minorée par une constante > 0 sur $\mathbb{S}^{m-1}$ si et seulement si (SK) est vérifiée. On retrouve ainsi le résultat de Shizuta et Kawashima, mais on peut également étudier les situations où (SK) n'est pas vérifiée.

En particulier, étudions la stabilité asymptotique dans L^2 du système (2.7). A la vue de (2.10), il est naturel d'introduire l'ensemble de dégénérescence

$$\mathcal{D} := \{\xi \in \mathbb{R}^m; N(\xi/|\xi|) = 0\},$$

qui est l'ensemble des points ξ pour lesquels 'la condition (SK) n'est pas vérifiée' (l'hypothèse (SK) correspond au cas où $\mathcal{D} = \{0\}$). On voit alors que la stabilité L^2 a lieu si et seulement si $\mathcal{D}$ est de mesure nulle. Or, l'utilisation de la condition de Kalman (issue de la théorie du contrôle) permet de montrer que l'ensemble de dégénérescence $\mathcal{D}$ est une variété algébrique de $\mathbb{R}^m$, donc seulement 2 situations peuvent se produire :

– ou bien $\mathcal{D}$ est de mesure nulle et alors il y a stabilité asymptotique dans L^2,
– ou bien $\mathcal{D} = \mathbb{R}^m$ et alors on peut montrer qu'il existe des solutions non dissipées.

Ainsi la CNS de stabilité asymptotique L^2 est mesure$(\mathcal{D}) = 0$, elle est strictement plus faible que (SK) en dimension $m > 1$.

Lorsque $\mathcal{D}$ est de mesure nulle, en étudiant finement la façon dont la fonction $\omega \in \mathbb{S}^{m-1} \mapsto N(\omega)$ s'annule au voisinage de $\mathcal{D}$, on peut obtenir une décomposition des solutions de (7.1) en 4 composantes $w = w_1 + w_2 + w_3 + w_4$ correspondant aux fréquences basses ou hautes, proches ou loin de $\mathcal{D}$. Les deux nouvelles composantes w_3 et w_4 décroissent moins vite que les 2 premières. Ainsi, en dimension $m \geqslant 2$, il existe toute une classe de phénomènes qui ne sont pas couverts par les travaux de Shizuta et Kawashima. Une telle décomposition est démontrée sous l'hypothèse supplémentaire que $\mathcal{D}$ est une union de sous-espaces vectoriels de $\mathbb{R}^m$. Dans le cas général, où $\mathcal{D}$ est une variété algébrique, la décomposition des solutions est un problème ouvert.

Résultats et techniques pour les systèmes non linéaires (2.8) : L'existence de solutions régulières globales, dans un voisinage d'un équilibre constant W_e (c'est-à-dire $Q(W_e) = 0$), a été démontrée par Yong [168], lorsque la condition (SK) est vérifiée sur le système linéarisé autour de cet équilibre et sous des hypothèses convenables d'entropie (voir également [98] pour le cas 1D). Les techniques développées ici pour les systèmes linéaires permettent de préciser ce résultat en explicitant un minorant pour la taille du voisinage dans lequel l'existence globale a lieu. Ce résultat est utile, par exemple, pour faire de la relaxation. Considérons un système partiellement dissipatif de la forme

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} \frac{\partial F_j(w)}{\partial x_j} = \frac{1}{\tau} Q(w)$$

où $\tau > 0$ est un paramètre de relaxation, destiné à tendre vers zéro. On a alors besoin que le problème de Cauchy soit bien posé pour des condition initiales appartenant à un ouvert uniforme par rapport à τ. Sous de bonnes hypothèses sur les fonctions $F_1, ..., F_m, Q$, cette existence uniforme par rapport à τ a bien lieu, et s'obtient comme corollaire du résultat précédent. Il s'agit d'une généralisation d'un résultat antérieur de Coulombel et Goudon [72].

2.7 Contrôlabilité de l'équation de Kolmogorov (A8) [34]

Modèle : Dans cet article, Enrique Zuazua et moi considérons l'équation de Kolmogorov,

$$\frac{\partial f}{\partial t}(t,x,v) + v\frac{\partial f}{\partial x}(t,x,v) - \frac{\partial^2 f}{\partial v^2}(t,x,v) = u(t,x,v)1_\omega(x,v), (x,v) \in \Omega, \tag{2.11}$$

où Ω est un ouvert de $\mathbb{R}^2$ et ω est un sous domaine de Ω. Il s'agit d'un système commandé dans lequel l'état est la fonction f et le contrôle est le terme source $u(t,x,v)$ localisé sur le sous-domaine ω.

Motivation : Le contrôle de l'équation de la chaleur

$$y_t(t,x) - \Delta y(t,x) = u(t,x)1_\omega(x), x \in \Omega,$$

(contrôlabilité à zéro, contrôlabilité approchée) est bien compris, pour des équations linéaires comme pour des équations semi-linéaires, en domaine borné comme en domaine non borné (voir, par exemple [76, 82, 84, 85, 86, 96, 141, 90, 126, 130, 115, 78, 79]. En particulier, l'équation de la chaleur, sur un domaine Ω borné, est contrôlable à zéro en tout temps $T > 0$ et avec un support ω arbitrairement petit pour le contrôle. De même, l'équation de la chaleur, sur $\mathbb{R}^n$, est contrôlable à zéro en tout temps $T > 0$, lorsque le support du contrôle ω est le complémentaire d'un compact dans $\mathbb{R}^n$.

L'équation de Kolmogorov diffuse dans les deux variables spatiales : elle diffuse en v grâce au terme $\partial_v^2 f$, mais également en x grâce à l'interaction entre le terme de transport $v\partial_x f$ et le terme de diffusion $\partial_v^2 f$. En fait, l'équation de Kolmogorov est un des exemples les plus simples de système hypocoercif [166]. Il est donc naturel de se demander si les résultats connus pour l'équation de la chaleur persistent pour cette équation, ou si des conditions géométriques sont nécessaires, comme pour l'équation des ondes. L'article [34] est un premier pas dans cette direction.

Résultats : On démontre la contrôlabilité à zéro de l'équation (2.11), dans deux configurations
- $\Omega_1 = \mathbb{R}_x \times \mathbb{R}_v$ et $\omega_1 = \mathbb{R}_x \times [\mathbb{R} - [a_1, b_1]]_v$, où $-\infty < a_1 < b_1 < +\infty$,
- $\Omega_2 = (0, 2\pi)_x \times (0, 2\pi)_v$ et $\omega_2 = (0, 2\pi)_x \times (a_2, b_2)_v$, où $0 < a_2 < b_2 < 2\pi$.

Dans le deuxième cas, les conditions imposées au bord du carré sont de type périodiques.

Techniques : La stratégie de preuve repose sur l'utilisation de la transformée de Fourier (continue pour Ω_1, série de Fourier pour Ω_2). Les deux preuves étant similaires je présente ici uniquement le traitement du premier cas.

Appliquons la transformée de Fourier en x à l'équation (2.11) : pour tout $\xi \in \mathbb{R}$, on obtient

$$\frac{\partial \hat{f}}{\partial t}(t, \xi, v) + i\xi v \hat{f}(t, \xi, v) - \frac{\partial^2 \hat{f}}{\partial v^2}(t, \xi, v) = \hat{u}(t, \xi, v) 1_{\mathbb{R}-[a_1,b_1]}(v), v \in \mathbb{R}^2. \tag{2.12}$$

Ainsi, pour tout $\xi \in \mathbb{R}$, la fonction $(t, v) \mapsto \hat{f}(t, \xi, v)$ résout une équation de la chaleur avec potentiel $i\xi v$. Une fois ce constat fait, la preuve de notre résultat repose sur deux ingrédients, comme dans [105] :
- un taux de décroissance exponentiel explicite pour la solution du système (2.12) avec $u = 0$,
$$\|\hat{f}(t, \xi, .)\|_{L^2(\mathbb{R}_v)} \leqslant \|\hat{f}_0(\xi, .)\|_{L^2(\mathbb{R}_v)} e^{-\xi^2 t^3/12}, \forall \xi \in \mathbb{R}, \forall t \in [0, +\infty),$$
obtenu en calculant explicitement la transformée de Fourier (en x et v de $f(t)$).
- un coût explicite (en ξ) pour la contrôlabilité à zéro de l'équation de la chaleur (2.12), de la forme $e^{C(T)\max\{1,\sqrt{|\xi|}\}}$, obtenu en démontrant une nouvelle estimation de Carleman.

La preuve de la contrôlabilité à zéro se fait alors en deux temps, selon le schéma classique : étant donné un temps $T > 0$,
- sur l'intervalle de temps $[0, T/2]$, on n'applique aucun contrôle ($u = 0$), afin de tirer profit de la dissipation naturelle du système
$$\|\hat{f}(T/2, \xi, .)\|_{L^2(\mathbb{R}_v)} \leqslant e^{-\xi^2 T^3/96}, \forall \xi \in \mathbb{R},$$
- sur l'intervalle de temps $[T/2, T]$, on utilise le contrôle $u(t, x, v)$ tel que $\hat{u}(t, \xi, v)$ réalise la contrôlabilité à zéro de $\hat{f}(T/2, \xi, v)$, pour tout $\xi \in \mathbb{R}$.

Il faut alors vérifier que le contrôle ainsi obtenu est bien dans $L^2(0,T)$, ce qui découle des majorations précédemment démontrées :

$$\begin{aligned}\int_{T/2}^T \int_{\mathbb{R}^2} |u(t,x,v)|^2 dx dv dt &= \int_{T/2}^T \int_{\mathbb{R}^2} |\hat{u}(t,\xi,v)|^2 d\xi dv dt \\ &\leqslant \int_{T/2}^T \int_{\mathbb{R}} e^{C(T)\max\{1,\sqrt{|\xi|}\}} \|\hat{f}(T/2,\xi,.)\|^2_{L^2(\mathbb{R}_v)} d\xi dt \\ &\leqslant \int_{T/2}^T \int_{\mathbb{R}} e^{C(T)\max\{1,\sqrt{|\xi|}\}} e^{-\xi^2 T^3/48} \|\hat{f}_0(\xi,.)\|^2_{L^2(\mathbb{R}_v)} d\xi dt \\ &\leqslant C\|f_0\|^2_{L^2(\mathbb{R}^2)} < +\infty.\end{aligned}$$

Nouveauté/Remarques : Cet article est un premier pas dans la compréhension de la contrôlabilité des systèmes hypocoercifs : nous proposons une méthode permettant de gérer une éventuelle hypocoercivité dans les arguments classiques (inégalités de Carleman). Ce résultat n'est cependant pas pleinement satisfaisant, puisque les configurations géométriques y sont particulières : le support de nos contrôles est invariant par translation selon l'axe des x, en raison de l'utilisation de la transformée de Fourier.

2.8 Contrôle en micromagnétisme (A7,P1) [10, 11]

Modèle : Cet article, co-écrit avec François Alouges, et ce proceeding, co-écrit avec François Alouges et Mario Sigalotti sont motivés par l'étude du retournement de l'aimantation dans les matériaux ferromagnétiques (par exemple, les MRAM). Le matériau est modélisé par un domaine Ω de $\mathbb{R}^2$ (couche mince) ou $\mathbb{R}^3$, son aimantation est un champ de vecteurs $m : \Omega \to \mathbb{S}^2$, dont l'évolution est régie par l'équation de Landau-Lifchitz

$$\begin{cases} \frac{\partial m}{\partial t} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m), \text{ dans } \Omega, \\ \frac{\partial m}{\partial \nu} = 0, \text{ sur } \partial\Omega, \end{cases} \tag{2.13}$$

où $\alpha > 0$ est un coefficient d'amortissement et $H(m)$ est le champ magnétique total,

$$H(m) = A\Delta m + H_d(m) + H_{ext}(t),$$

où $A > 0$ est la constante d'échange, $H_d(m)$ est le champ démagnétisant, et $H_{ext}(t)$ est le champ magnétique extérieur, uniforme en espace. Le champ démagnétisant est lui même défini de la façon suivante

$$\begin{cases} H_d(m) = \nabla\phi \text{ dans } \mathbb{R}^3, \\ \Delta\phi = -\text{div}(\overline{m}) \text{ dans } \mathbb{R}^3, \\ H_d(m) \text{ s'annule à l'infini,} \end{cases}$$

où $\overline{m}(x) = m(x)$ si $x \in \Omega$ et $\overline{m}(x) = 0$ sinon. Une autre interprétation consiste à dire que $H_d(m)$ est la projection orthogonale, pour le produit scalaire $L^2(\mathbb{R}^3)$, du vecteur $-\overline{m}$, sur l'espace vectoriel engendré par les gradients. L'opérateur H_d est donc non local. Une énergie est naturellement associée à ce système,

$$\mathcal{E}(m) := \frac{A}{2}\int_\Omega |\nabla m|^2 - \frac{1}{2}\int_\Omega \langle H_d(m), m\rangle.$$

L'équation de Landau-Lifchitz (2.13) est un système commandé non linéaire dans lequel l'état est l'aimantation m et le contrôle est le champ magnétique extérieur $H_{ext}(t)$. Notre but est d'étudier le problème suivant : étant donné un minimiseur global u de l'énergie $\mathcal{E}$, existe-t-il un temps $T > 0$ et un contrôle $H_{ext} : [0,T] \to \mathbb{R}^3$ tel que la solution de l'équation de Landau-Lifchitz (2.13) de condition initiale $m(0) = u$ existe pour tout $t \in [0,T]$ et vérifie $m(T) = -u$. Dans le procedding [11], on se pose cette question, en imposant différentes contraintes supplémentaires sur le champ extérieur, notamment :

- d'être 2D (comme c'est le cas dans les MRAM) ou
- d'être de la forme $H_{ext}(t,x) = h(t)m(t,x) \wedge e$ où e est un vecteur normé et $h : [0,T] \to \mathbb{R}$ est le nouveau contrôle (modèle d'injection de spin).

Difficultés liées au modèle : L'équation de Landau-Lifchitz est un modèle délicat à manipuler, pour de multiples raisons. La première est que le modèle sous-jacent (obtenu en négligeant le terme gyroscopique, le champ démagnétisant et le champ extérieur) est le flot des applications harmoniques

$$\frac{\partial m}{\partial t} = \Delta m + |\nabla m|^2 m,$$

équation qui présente déjà des pathologies : ses solutions faibles ne sont pas uniques, l'existence de solutions fortes (qui, elles, sont uniques) n'est établie que sous des hypothèses restrictives, par exemple, une énergie initiale petite, ou une condition initiale à valeurs dans la demi-sphère, parce que des phénomènes d'explosion en temps fini sont possibles (voir, par exemple [59], [70]). De plus, on y ajoute un opérateur non local $H_d(.)$ et un terme gyroscopique.

Domaines ellipsoïdaux : Lorsque que Ω est une ellipsoïde, le champ démagnétisant d'une aimantation constante (en espace) est également constant : il existe une matrice symétrique $D \geqslant 0$ telle que $H_d(m) = -Dm, \forall m \in \mathbb{S}^2$. On peut supposer que

$$D := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \text{ où } 0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3.$$

Les domaines ellipsoïdaux sont donc un cas particulier où la non localité de l'opérateur H_d est moins violente que dans le cas général. Une sous-classe de solutions de l'équation (2.13) est alors constituée des solutions de l'équation différentielle

$$\frac{dm}{dt} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m) \text{ où } H(m) := -Dm + H_{ext}(t). \tag{2.14}$$

On peut également montrer (par un argument de rescaling) que, si l'ellipsoïde est assez petite, alors $\pm e_1$ sont des minimiseurs globaux de l'énergie $\mathcal{E}$.

Résultats et techniques : Pour l'équation différentielle (2.14), on démontre les résultats suivants :

- avec des commandes $H_{ext}(t)$ 3D, le problème de contrôle est essentiellement trivial : on peut suivre n'importe quelle trajectoire $m_{ref}(t)$, point par point, en choisissant convenablement la commande $H_{ext}(t)$; on étudie alors les commandes optimales (c'est-à-dire dont la norme $L^2(0,T)$ est minimale) pour le retournement de $+e_1$ à $-e_1$,
- avec des commandes $H_{ext}(t)$ 2D, nous caractérisons les trajectoires réalisables et nous en déduisons que le retournement de $+e_1$ à $-e_1$ est possible,
- pour le modèle d'injection de spin, nous donnons une condition nécessaire et suffisante sur le vecteur e pour que le système soit contrôlable, nous en déduisons alors que, sous cette hypothèse, le retournement de $+e_1$ à $-e_1$ est possible.

Pour l'EDP (2.13), on commence par étudier les solutions faibles. On démontre l'existence de solutions faibles globales, en adaptant une preuve antérieure de Alouges et Soyeur [12]. Puis on étudie la convergence de ces solutions faibles vers des aimantations uniformes, lorsque la taille de l'ellipsoïde tend vers zero. Ceci justifie que les contrôles précédemment trouvés (pour l'EDO), réalisent, de façon approchée, le retournement d'aimantation, sur des domaines suffisamment petits.

On étudie ensuite les solutions fortes de l'EDP (2.13). Précisemment, on démontre

- l'existence et l'unicité de solutions fortes locales (en temps) lorsque Ω est un domaine borné de $\mathbb{R}^2$ ou $\mathbb{R}^3$,
- que ces solutions sont globales (en temps) lorsque $\Omega \subset \mathbb{R}^2$ et que la condition initiale m_0 est dans un voisinage H^1 des constantes,
- que ces solutions sont globales (en temps) lorsque Ω est une ellipsoide 3D et que la condition initiale est dans un voisinage H^2 des constantes.

La preuve des deux premiers points est une adaptation d'une preuve antérieure de Carbou et Fabrie [50], mais le troisième point utilise des arguments différents.

Enfin, on aborde le problème du retournement de l'aimantation sur l'EDP. On ne le fait que sur de petites ellipsoides 2D ou 3D parce que, dans ce cas, on dispose de solutions fortes globales, et on connait explicitement les minimiseurs globaux de l'energie ($\pm e_1$). On propose des contrôles (en boucle ouverte) explicites qui, pour des conditions initiales m_0 suffisamment proches de e_1 dans H^2 garantissent l'existence d'une solution forte globale, qui converge exponentiellement vite vers $-e_1$ quand $t \to +\infty$. Ces contrôles sont les suivants :

- sur un intervalle borné $[0, T]$, on applique un contrôle qui retourne exactement l'EDO de $+e_1$, à $-e_1$; pour des raisons de continuité par rapport aux conditions initiales le même controle amène la solution de l'EDP d'une condition initiale proche de e_1 dans H^2 à une valeur $m(T)$ proche de $-e_1$ dans H^2.
- sur l'intervalle $[T, +\infty)$, on n'applique aucun contrôle ; la dissipation naturelle du système assure alors la convergence exponentielle vers $-e_1$.

Deuxième partie

Activité de recherche détaillée

In this habilitation I worked on several control problems for PDEs, coming from different fields (quantum mechanics, kinetic equations, micromagnetism) and presenting different pathologies (bilinear control in quantum control, hypoellipticity for kinetic equations, pathologies of the heat flow of harmonic maps in micromagnetism). I also tried to use classical tools from control theory (Kalman rank condition) in order to solve more classical problems of PDE analysis (asymptotic behavior, well posedness).

The structure of this part is the following one.

The Chapter 3 deals with exact controllability for bilinear control systems with discrete spectrum.

The Chapter 4 is dedicated to feedback stabilization of bilinear Schrödinger equations.

In Chapter 5, we focus on a linear multi-D Schrödinger equation (derived from a bilinear system by linearization) and we study the genericity with respect to the spacial domain of its spectral controllability.

The Chapter 6 is devoted to the Bloch equation, which is an infinite dimensional bilinear control system, with continuous spectrum, taking the form of a continuum of ODEs (instead of a PDE). We study its controllability and its stabilization.

The Chapter 7 deals with partially dissipative hyperbolic systems : we study the asymptotic behavior in the linear case and the existence of global smooth solutions in the nonlinear case.

In Chapter 8, we present a controllability result for the Kolmogorov equation, which is an hypocoercive linear kinetic equation.

Finally, Chapter 9 deals with the analysis and the control of the Landau-Lifschitz equation, from micromagnetism.

Chapters 3 to 5 are natural continuations of my PHD work, Chapters 6 to 9 deal with new subjects or new models.

The stucture of each chapter is the same : first, we present the context and the bibliography, then, we present the new contributions, finally, we propose open problems and perspectives.

Chapitre 3

Exact Controllability of bilinear systems with discrete spectrum (A4, A5, A12, A13) [23, 24, 31, 25]

This Chapter is organized as follows. In Section 3.1, we propose a review of known controllability results for bilinear control systems, both in finite dimension and infinite dimension. In Section 3.2, we present the main technics used in this chapter : linearization principle, moment method, Nash-Moser theorem and Coron's return method. In Section 3.3, we present the article [23] **(A4)**, dealing with a 1D Schrödinger equation on a variable domain. In Section 3.4, we present the article [24] **(A5)**, about a 1D beam equation. In Section 3.5, we present the article [25] **(A13)**, dealing with a 1D wave equation. In Section 3.6, we present the article [31] **(A12)** about linear and nonlinear Schrodinger equations.

3.1 Context, review of previous results

The controllability of PDEs with distributed and boundary controls, acting **linearly** on the state, is studied since a long time. Let us give a brief bibliography concerning the Schrödinger or wave equations. For linear equations, thanks to the Hilbert Uniqueness method, the controllability is equivalent to an observability inequality that may be proved with different technics : multiplier methods (see [81] by Fabre, [127] by Machtyngier), microlocal analysis (see [122] by Lebeau, [46] by Burq), Carleman estimates (see [118, 119] by Lasiecka, Triggiani, Zhang), or number theory (see [144] by Ramdani, Takahashi, Tenenbaum and Tucsnak). For nonlinear equations, we refer to [74] by Dehman, Gérard, Lebeau, [117] by Lange, Teismann, [120, 121] by Laurent, [147] by Rosier, Zhang.

The study of the controllability of PDEs with **bilinear** controls started later. One of the reasons of this delay may be a negative result proved by Ball, Marsden and Slemrod [17] (presented in Subsection 3.1.3) : because of this non controllability result, bilinear systems have been considered as non controllable for a long time. However, progress have been made in the last years and this question is now understood in a better way.

In this section, we give a review of known controllability results for bilinear systems. The

Subsection 3.1.1 deals with exact controllability of finite dimensional control systems, i.e. ordinary differential equations. The following subsections are devoted to infinite dimensional systems, i.e. PDEs. In Subsection 3.1.2, we explain why geometric methods are not as powerful in infinite dimension than in finite dimension. In Subsection 3.1.3, we recall Ball, Marsden and Slemrod's negative result. In Subsection 3.1.4, I recall the exact controllability results proved in my PHD thesis. In Subsection 3.1.5, we present other results about bilinear systems, not dealing with exact controllability, but rather with approximate controllability and optimal control.

3.1.1 Controllability of finite dimensional bilinear systems : the Lie rank condition

Let us consider the nonlinear control system

$$\dot{x} = f(x, u) \tag{3.1}$$

where $x \in \mathbb{C}^n$ is the state, $u \in \mathbb{R}^m$ is the control, with $(x, u) \in \mathcal{O}'$ where $\mathcal{O}'$ is a nonempty open subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 1 *Let (x_e, u_e) be an equilibrium of the control system (3.1) (i.e. $(x_e, u_e) \in \mathcal{O}'$ and $f(x_e, u_e) = 0$). The system (3.1) is small time locally controllable at the equilibrium (x_e, u_e) if, for every $\epsilon > 0$, there exists $\eta > 0$ such that, for every $x^0 \in \mathbb{R}^n$ with $\|x^0 - x_e\| < \eta$ and for every $x^1 \in \mathbb{R}^n$ with $\|x^1 - x_e\| < \eta$, there exists a measurable function $u : [0, \epsilon] \to \mathbb{R}^m$ such that $\|u - u_e\|_{L^\infty(0,\epsilon)} < \epsilon$ and the solution of (3.1) with initial condition $x(0) = x^0$ satisfies $x(\epsilon) = x^1$.*

One does not know any checkable necessary and sufficient condition for small time local controllability for general control systems, even for analytic control systems. However, one knows powerful necessary conditions and powerful sufficient conditions, for example

- the necessary Lie algebra rank condition, that relies on iterated Lie brackets,
- in the case of driftless control affine systems, this condition turns out to be sufficient, even for global controllability.

We present them in the next paragraphs.

Iterated Lie brackets and Lie algebra condition

Definition 2 *Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^n$ and $X = (X^1, ..., X^n)^t, Y = (Y_1, ..., Y^n) \in C^1(\mathcal{O}, \mathbb{R}^n)$. The* ***Lie bracket*** *$[X, Y] := ([X, Y]^1, ..., [X, Y]^n)^t$ of X and Y is the element in $C^0(\mathcal{O}, \mathbb{R}^n)$ defined by*

$$[X, Y](x) := Y'(x)X(x) - X'(x)Y(x), \forall x \in \mathcal{O}.$$

In other words, the components of $[X, Y](x)$ are

$$[X, Y]^j(x) = \sum_{k=1}^{n} \left(X^k(x) \frac{\partial Y^j}{\partial x_k}(x) - Y^k(x) \frac{\partial X^j}{\partial x_k}(x) \right), \forall j \in \{1, ..., n\}, \forall x \in \mathcal{O}.$$

Definition 3 *Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^n$ and let $\mathcal{F}$ be a family of vector fields of class C^∞ in $\mathcal{O}$. We denote by Lie($\mathcal{F}$) the* ***Lie algebra generated by the vector fields in*** *$\mathcal{F}$, i.e., the smallest linear subspace E of $C^\infty(\mathcal{O},\mathbb{R}^n)$ satisfying*

$$\mathcal{F} \subset E \tag{3.2}$$

$$(X \in E \text{ and } Y \in E) \Rightarrow ([X,Y] \in E). \tag{3.3}$$

(such a smallest subspace does exist : $E = C^\infty(\mathcal{O},\mathbb{R}^n)$ satisfies (3.2) and (3.3), consider the intersection of all the linear subspaces of $C^\infty(\mathcal{O},\mathbb{R}^n)$ satisfying (3.2) and (3.3)).

With these definitions, one has the following well known necessary condition for the small time local controllability of analytic control systems, due to Hermann [101] and to Tadashi and Nagano[135] (see also [156] by Sussmann).

Theorem 1 *Assume that the control system (3.1) is small time locally controllable at the equilibrium point (x_e, u_e) and that f is analytic. Then,*

$$\mathcal{A}(x_e,u_e) := \Big\{ g(x_e); g \in Lie\Big(\frac{\partial^{|\alpha|} f}{\partial u^\alpha}(.,u_e), \alpha \in \mathbb{N}^n\Big)\Big\} = \mathbb{R}^n. \tag{3.4}$$

This necessary condition for small time local controllability is not sufficient in general, but it turns out to be sufficient for driftless control affine systems, that is, in the case $f(x,u) = \sum_{i=1}^m u_i f_i(x)$. This is the classical Rashevski-Chow theorem (see [145, 58]) that we recall in the following paragraph.

Control affine systems

For driftless control affine systems, one also have a global controllability criterium, proved independently by Rashevski [145] and Chow [58].

Theorem 2 *Let $\mathcal{O}$ be a connected nonempty open subset of $\mathbb{R}^n$, $\mathcal{O}' := \mathcal{O} \times \mathbb{R}^m$, $f_1, ..., f_m \in C^\infty(\mathcal{O},\mathbb{R}^n)$ and $f(x,u) := \sum_{i=1}^m u_i f_i(x)$.*

- *Let $x_e \in \mathcal{O}$ be such that $\mathcal{A}(x_e, 0) = \mathbb{R}^n$, where $\mathcal{A}(x_e, 0)$ is defined by (3.4). Then, the control system $\dot{x} = \sum_{i=1}^m u_i f_i(x)$ is small time locally controllable at $(x_e, 0)$.*
- *If $\mathcal{A}(x, 0) = \mathbb{R}^n$, $\forall x \in \Omega$, then for every $(x^0, x^1) \in \Omega \times \Omega$ and for every $T > 0$ there exists $u \in L^\infty((0,T),\mathbb{R}^m)$ such that the solution of*

$$\begin{cases} \dot{x} = \sum_{i=1}^m u_i f_i(x), \\ x(0) = x^0, \end{cases}$$

is defined on $[0,T]$ and satisfies $x(T) = x^1$.

We also have the following result for control affine systems

$$\begin{cases} \dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x), \\ x(0) = x^0 \end{cases} \tag{3.5}$$

(see [65, Corollary 3.26 page 141] for a proof).

Theorem 3 *Let $\mathcal{O}$ be a connected nonempty open subset of $\mathbb{R}^n$ and $f_0, f_1, ..., f_m \in C^\infty(\mathcal{O}, \mathbb{R}^n)$. We assume there exists $k \in \{1, ..., n\}$ such that, for every $x \in \mathcal{O}$,*

$$Span\{h(x); h \in Lie(f_0, f_1, ..., f_m)\} \text{ is of dimension } k.$$

Then, for every $x^0 \in \mathcal{O}$, the reachable set from x^0,

$$\left\{x^1 \in \mathbb{R}^n; \exists T > 0, \exists u \in L^\infty((0,T), \mathbb{R}^m) / \text{the solution of } (3.5) \text{ satisfies } x(T) = x^1\right\}$$

is contained in a submanifold of $\mathcal{O}$ of dimension k.

For a more detailed study of the controllability of finite dimensional nonlinear control systems, we refer to [65, Chapter 3]. Notice that there exists many other sufficient conditions for the small time local controllability of (3.1), under different assumptions (see, for example [3, 5, 6, 36, 37, 108, 159] and references cited in these papers).

Bilinear Schrödinger systems

This subsection is dedicated to bilinear Schrödinger finite dimensional systems,

$$i\frac{dX}{dt} = H_0 X + u(t) H_1 X, \tag{3.6}$$

where H_0 and H_1 are $n * n$ hermitian matrices with complex coefficients, $X : (0,T) \mapsto \mathbb{C}^n$ is the state and $u : (0,T) \mapsto \mathbb{R}$ is the control.

The controllability of such systems has been studied with different tools (see for example [8, 13, 44, 142, 157, 161]). An important necessary and sufficient condition for the global controllability was proved by Albertini and D'Alessandro in [8].

Theorem 4 *The system (3.6) is globally controllable if and only if $Lie(H_0, H_1)$ is isomorphic (conjugated) to*

- *$su(n)$ if n is odd,*
- *$su(n)$ or $sp(n/2)$ if n is even.*

We also refer to the following works about the controllability of finite dimensional quantum systems [4, 14, 38, 39, 40, 41, 73, 42, 43], by Agrachev, Boscain, Chambrion, Charlot, Gauthier, Guérin, Jauslin and Mason, [109] by Khaneja, Glaser and Brockett, [143] by Ramakrishna, Salapaka, Dahleh, Rabitz, [157] by Sussmann and Jurdjevic, [161] by Turinici and Rabitz. The books [107, 7] may be useful for the study of these systems. Let us also mention [134] by Mirrahimi, Rouchon, Turinici and [28] for explicit feedback controls, inspired by Lyapunov technics.

3.1.2 Iterated Lie brackets in infinite dimension

In this subsection, we explain why iterated Lie brackets are less powerful in infinite dimension than in finite dimension.

A favorable example

First, let us show, on an example, how iterated Lie brackets can sometimes be useful for studying the controllability in infinite dimension. This example is borrowed from [133]. We consider the following system

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + x^2\psi - u(t)x\psi, x \in \mathbb{R}, t \in (0,T). \tag{3.7}$$

It is a control system in which

- the state is the wave function ψ with

$$\int_{\mathbb{R}} |\psi(t,x)|^2 dx = 1, \forall t \in [0,T],$$

- the control is the real valued function $u : [0,T] \to \mathbb{R}$, which corresponds to a classical electro-magnetic field.

The free Hamiltonian

$$H_0(\psi) := -\psi'' + x^2\psi$$

corresponds to the usual harmonic oscillator. With our previous notations, we define the vector fields

$$f_0(\psi) := -\psi'' + x^2\psi,$$
$$f_1(\psi) := x\psi.$$

At a formal level, we have

$$\begin{aligned}&[f_0, f_1](\psi) = -2\psi',\\ &[f_0, [f_0, f_1]](\psi) = 4x\psi = 4f_1(\psi),\\ &[f_1, [f_0, f_1]](\psi) = 2\psi.\end{aligned}$$

Thus, the Lie algebra generated by f_0 and f_1 is of dimension 4 : it is the linear space generated by f_0, f_1, $[f_0, f_1]$ and Id. Hence, by Theorem 3, one would expect that the dimension of the reachable set from a given point should be of dimension at most 4. However, this Theorem cannot be applied because we are in infinite dimension and, moreover, f_0 and f_1 are not smooth and not even defined on the whole $L^2(\mathbb{R}, \mathbb{C})$-sphere. But one can check, by direct computations, that the intuition given by the above arguments is indeed correct. We present an heuristic of these computations and we refer to [133] for precise arguments. Let $p, q : [0,T] \to \mathbb{R}$ be defined by

$$q(t) := \int_{\mathbb{R}} x|\psi(t,x)|^2 dx,$$

$$p(t) := -2\Im \int_{\mathbb{R}} \overline{\psi'}\psi.$$

Note that $q(t)$ is the average position and $p(t)$ is the average momentum of the quantum system. The dynamics of the couple (p,q) is given by

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -4q + 2u. \end{cases}$$

By the Kalman rank condition (see, for example, [65, Theorem 1.16 page 9] or or [158, Theorem 2.2.1]) this linear control system is globally controllable. Let us define $\phi \in C^0([0,T],\mathbb{S})$ ($\mathbb{S}$ is the $L^2(\mathbb{R},\mathbb{C})$-sphere) by

$$\phi(t,x) := \psi(t,x+q)e^{-i\frac{p}{2}x+ir}$$

where

$$r(t) := \int_0^t \Big(q(s)^2 - \frac{3}{4}p(s)^2 - u(s)q(s)\Big)ds.$$

Then, straightforward computations lead to

$$\frac{\partial \phi}{\partial t} = -\frac{\partial^2 \phi}{\partial x^2} + x^2\phi,$$

thus the evolution of ϕ does not depend on the control u. This allows to share the state ψ into two parts

- a 2 dimensional controllable part (p,q), that corresponds to the classical dynamics of the particle (average position q and average momentum p),
- an infinite dimensional non controllable part ϕ.

With this precise description of ψ, one may check that, given $\psi_0 \in \mathbb{S}$, the reachable set

$$\Big\{\psi(T); T > 0, u : (0,T) \to \mathbb{R}, \psi \text{ solution of (3.7) such that } \psi(0) = \psi_0\Big\}$$

is contained in a submanifold of $\mathbb{S}$ of dimension 4.

An unfavorable example

Now, let us show, on an example, why iterated Lie brackets may sometimes be less powerful in infinite dimension than in finite dimension. We consider the following Schrödinger equation

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - u(t)x^2\psi, x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0. \end{cases} \tag{3.8}$$

It is a control system in which

- the state is the wave function ψ with

$$\int_0^1 |\psi(t,x)|^2 dx = 1, \forall t \in [0,T],$$

- the control is the function $u : [0,T] \to \mathbb{R}$, which corresponds to a classical electromagnetic field.

In Section 3.6, we will see that this control system is controllable in $H^3((0,1),\mathbb{C})$, in any positive time $T > 0$, with controls in $L^2((0,T),\mathbb{R})$, locally around the ground state $\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$. Let us define the operators f_0 and f_1

$$\begin{array}{ll} D(f_0) := H^2 \cap H^1_0((0,1),\mathbb{C}) & f_0(\psi) := -\psi'', \\ D(f_1) := L^2((0,1),\mathbb{C}) & f_1(\psi) := x^2\psi. \end{array}$$

Let us compute the iterated Lie bracket at the point $\varphi_1(x) := \sqrt{2}\sin(\pi x)$. Since $\varphi_1 \in D(f_0)$, we can compute

$$\begin{array}{c} [f_0,f_1](\varphi_1) = -4x\varphi_1' - 2\varphi_1, \\ [f_1,[f_0,f_1]](\psi) = 8x^2\varphi_1 = 8f_1(\varphi_1). \end{array}$$

Notice that $[f_0, f_1](\varphi_1)$ does not belong to $D(f_0)$ because $[f_0, f_1](\varphi_1)(1) = 4\sqrt{2}\pi \neq 0$. Thus, in order to give a sense to the Lie bracket $[f_0, [f_0, f_1]]$, one needs to extend the definition of f_0 to functions that do not vanish at $x = 0, 1$. A natural choice is

$$f_0(\psi) := -\psi'' + \psi(0)\delta_0' - \psi(1)\delta_1' \tag{3.9}$$

because, with this choice, we have

$$\langle f_0(\psi), \widetilde{\psi}\rangle = \langle \psi, f_0(\widetilde{\psi})\rangle, \forall \psi \in D(f_0), \forall \widetilde{\psi} \in H^2((0,1), \mathbb{C}),$$

in the sense

$$-\int_0^1 \psi''(x)\widetilde{\psi}(x)dx = -\int_0^1 \psi(x)\widetilde{\psi}''(x)dx - \psi'(1)\widetilde{\psi}(1) + \psi'(0)\widetilde{\psi}(0),$$

which is an integration by parts. With the definition (3.9), we get

$$[f_0, [f_0, f_1]](\psi) = -8 f_0(\psi) + 4\psi'(1)\delta_1'$$

But then, again, $[f_0, [f_0, [f_0, f_1]]]$ is not well defined. Moreover, even if we could give a sense to any iterated Lie bracket, because of the presence of Dirac masses, it would not be clear which space the Lie algebra should generate in case of local controllability. Therefore, the way the Lie algebra rank condition could be used directly in infinite dimension is not clear.

Lie brackets and Galerkin approximations

Another natural strategy consists in

- first, discretizing the PDE or considering its Galerkin approximations, in order to get a system with finite dimension N,
- proving the exact controllability of this finite dimensional system thanks to geometric control methods,
- trying to pass to the limit as $N \to +\infty$.

This strategy has been used successfully by Sarychev and Agrachev [2] and by Shirikyan [150], to prove the exact controllability of dissipative equations, by Boscain, Chambrion, Mason and Sigalotti [54] to prove the approximate controllability in L^2, of bilinear Schrödinger equations.

At the present time, this strategy does not seem to be successful to prove exact controllability for non dissipative PDEs (such as Schrödinger equation). Moreover, one has to be careful with this approach. Indeed, let us consider the following modal approximation of the control system (3.7)

$$\frac{dX}{dt} = H_0 X + u(t) H_1 X$$

where

$$H_0 := \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{3}{2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{2n+1}{2} \end{pmatrix},$$

$$H_1 := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & \dots & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{n+1} \\ 0 & 0 & 0 & 0 & \dots & \sqrt{n+1} & 0 \end{pmatrix}.$$

It is a control system in which the state is $X \in \mathbb{S}_{2n+1} := \{Z \in \mathbb{C}^n; \|Z\| = 1\}$ and the control is u. By applying D'Alessandro's Theorem 4, Fu, Schirmer and Solomon proved in [87] that, for every $n \in \mathbb{N}$, this Galerkin approximation is globally controllable in large time : for every $X_0, X_1 \in \mathbb{S}_{2n+1}$, there exists $T > 0$ and $u \in L^\infty(0,T)$ such that the solution of the Cauchy problem

$$\begin{cases} \frac{dX}{dt} = H_0 X + u(t) H_1 X, \\ X(0) = X_0, \end{cases}$$

satisfies $X(T) = X_1$. Therefore, in the case of the example (3.7)

- the infinite dimensional system is strongly not controllable,
- but any Galerkin approximation is controllable.

This result shows that one has to be careful with finite dimensional approximations in order to get controllability of infinite dimensional systems. One also has to be careful with the hints given by 'formal' Lie brackets in infinite dimension.

In conclusion, for infinite dimensional systems, there are cases where the iterated Lie brackets provide the right intuition, but it is not always the case. This motivates the search of different methods for the study of the exact controllability of bilinear PDEs.

3.1.3 Ball, Marsden and Slemrod's negative result

The first controllability result for infinite dimensional bilinear systems is negative. It is proved by Ball, Marsden and Slemrod [17] and presented in this subsection. Because of this non controllability result, such equations have been considered as non controllable for a long time. Let us also mention that this negative result has been adapted to linear Schrödinger equation by Turinici in [160] and to nonlinear Schrödinger equations by Ilner, Lange and Teismann in [104].

In [17], Ball, Marsden and Slemrod discuss the controllability of infinite dimensional bilinear control systems of the form

$$\dot{w}(t) = \mathcal{A}w(t) + p(t)\mathcal{B}(w(t)), \tag{3.10}$$

where the state is w and the control is p. Thanks to Baire lemma, they prove the following non controllability result.

Theorem 5 *Let X be a Banach space with dim$(X) = +\infty$. Let $\mathcal{A}$ generate a C^0-semi group of bounded linear operators on X and $\mathcal{B} : X \to X$ be a bounded linear operator. Let $w_0 \in X$ be fixed and let $w(t; p, w_0)$ denote the unique solution of (3.10) for $p \in L^1_{loc}((0,+\infty),\mathbb{R})$ with $w(0) = w_0$. The set of states accessible from w_0, defined by*

$$\mathcal{R}(w_0) := \{w(t; p, w_0); t \geqslant 0, p \in L^r_{loc}((0,\infty),\mathbb{R}), r > 1\},$$

is contained in a countable union of compact subsets of X and, in particular, it has an empty interior in X.

For the proof of this theorem, we refer to [17] or [65, Theorem 9.4, page 248] for the Hilbert case. As noticed by G. Turinici in [160], a consequence of Theorem 5 is the following negative controllability result for the system

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\Delta\psi(t,x) - u(t)\mu(x)\psi(t,x), x \in \Omega, t \in (0,+\infty), \\ \psi(t,x) = 0, x \in \partial\Omega, \end{cases} \tag{3.11}$$

where Ω be a bounded open subset of R^N, $N \in \{1,2,3\}$, $\mu \in W^{2,\infty}(\Omega,\mathbb{R}^N)$, the state is ψ, $\psi(t) \in \mathbb{S}$ (the $L^2(\Omega,\mathbb{C})$-sphere) for every t and the control is the real valued function u.

Theorem 6 *The system (3.11) is not controllable in $H^2 \cap H_0^1(\Omega,\mathbb{C})$ with control functions in $L^r_{loc}(\mathbb{R}_+,\mathbb{R}^N)$, $r > 1$: for every $\psi_0 \in \mathbb{S} \cap H^2 \cap H_0^1(\Omega,\mathbb{C})$, the attainable set*

$$\mathcal{R}(\psi_0) := \{\psi(T;u,\psi_0); T > 0, u \in L^r((0,T),\mathbb{R}^N), r > 1\}$$

has a dense complement in $\mathbb{S} \cap H^2 \cap H_0^1(\Omega,\mathbb{C})$.

However, there is no obstruction for having controllability in different spaces. For example, let us consider the dipolar moment $\mu(x) = x^2$ on the domain $\Omega := (0,1)$ of $\mathbb{R}$. Then the Theorem 5 does not apply with

$$\widetilde{X} := H^3_{(0)}((0,1),\mathbb{C}) := \{\varphi \in H^3((0,1),\mathbb{C}); \varphi = \varphi'' = 0 \text{ at } x = 0,1\}$$

instead of X. Indeed,

- the operator A defined by

$$D(A) := H^2 \cap H_0^1((0,1),\mathbb{C}), \quad A\varphi = -\varphi'' \tag{3.12}$$

  generates a C^0-semi-group of bounded linear operators of $\widetilde{X}$
- for $\varphi \in \widetilde{X}$, $\mu\varphi$ belongs to $H^3 \cap H_0^1((0,1),\mathbb{R})$,
- but $(\mu\varphi)''$ coincides with $2\mu'\varphi'$ at $x = 0,1$, which is, in general, different from zero. Thus, $\varphi \mapsto \mu\varphi$ does not map $\widetilde{X}$ into $\widetilde{X}$.

Notice that Theorem 5 does not apply neither with

$$\overline{X} := H^3 \cap H_0^1((0,1),\mathbb{R})$$

(which is a space such that $\varphi \mapsto \mu\varphi$ maps $\overline{X}$ into $\overline{X}$) instead of X, because A does not generate a C^0 semi-group of bounded operators of $\overline{X}$.

In Section 3.6, we will see that, when $N = 1$ and μ is well chosen, then the system (3.11) is indeed exactly controllable in $H^3_{(0)}((0,1),\mathbb{C})$, with $L^2((0,T),\mathbb{R})$ controls, locally around the ground state. Thus, in this case, the negative result proved by Ball, Marsden, Slemrod and by Turinici is only due to a choice of functional spaces that does not allow exact controllability. In particular, it is not related to a deep non controllability, as, for example, when a subsystem evolves independently of the control.

3.1.4 First positive exact controllability results in infinite dimension

Concerning exact controllability issues, local results for 1D models have been proved in [22] **(A1)**, almost global results have been proved in [27] **(A2)**, during my PHD thesis. These results are presented in details here below, in order to explain the progresses made in this habilitation. In [64], Coron also proved that a positive minimal time is required for the local exact controllability of these 1D models.

Presentation of the article (A1) [22] :

Equation and result : The article [22] deals with the Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) - u(t)x\psi(t,x), x \in (-1/2,1/2), \\ \psi(t,\pm 1/2) = 0, \end{cases} \tag{3.13}$$

that represents a quantum particle in an infinite square potential well, in an electric field u. It is a bilinear control system in which the state is ψ, with $\psi(t) \in \mathbb{S}$ (the $L^2((-1/2,1/2),\mathbb{C})$-sphere) for every t, and the control is the real valued function u.

First, let us introduce few notations. Let A_1 be the operator defined by

$$D(A_1) := H^2 \cap H^1_0((-1/2,1/2),\mathbb{C}), \quad A_1\varphi := -\frac{d\varphi}{dx^2}.$$

Let us introduce its eigenvalues $\lambda_k := (k\pi)^2, \forall k \in \mathbb{N}^*$, the associated normalized eigenvectors $(\varphi_k)_{k\in\mathbb{N}^*}$, and the spaces $H^s_{(0)} := D(A_1^{s/2})$, for $s \in \mathbb{N}$.

In [22], we prove the local exact controllability of the system (3.13) in an $H^7_{(0)}$-neighborhood of the ground state $\varphi_1(x)e^{-i\lambda_1 t}$.

Theorem 7 *Let $\phi_0, \phi_1 \in \mathbb{R}$. There exists $T > 0$ and $\eta > 0$ such that, for every $\psi_0, \psi_f \in \mathbb{S} \cap H^7_{(0)}$ satisfying*

$$\|\psi_0 - \varphi_1 e^{i\phi_0}\|_{H^7} < \eta, \|\psi_f - \varphi_1 e^{i\phi_1}\|_{H^7} < \eta,$$

there exists a trajectory (ψ, u) of (3.13) on $[0,T]$ such that $\psi(0) = \psi_0$, $\psi(T) = \psi_f$ and $u \in H^1_0((0,T),\mathbb{R})$.

Interest : This result was the first positive exact controllability result for an infinite dimensional bilinear control system. It highlights the importance of the choice of the functional spaces in the proof of controllability results. Indeed, for the particular example studied : Ball, Marsden and Slemrod's theorem (or rather Turinici's corollary) provides the non exact controllability in $H^2_{(0)}$, but the exact controllability holds (at least locally) in $H^7_{(0)}$.

Another interest of this work is the introduction of Nash-Moser technics in control theory.

Technics : The technics used in this article are the linearization principle, Coron's return method, the Nash-Moser theorem, the moment method and quasi-static deformations that are described in Subsection 3.2.

Presentation of the article (A2) [27] :

Equation and result : The article [27] deals with the system

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) - u(t)x\psi(t,x), x \in (-1/2,1/2), \\ \psi(t,\pm 1/2) = 0, \\ \frac{dD}{dt}(t) = S(t), \\ \frac{dS}{dt}(t) = u(t). \end{cases} \tag{3.14}$$

This systems represents a quantum particle in an moving infinite square potential well (the 'box'). The variable D (resp. S, resp. u) is the position (resp. speed, resp. acceleration) of the potential well. It is a control system in which the state is (ψ, S, D) and the control is u : one wants to control the wave function of the particle, the position and the speed of the box, through the acceleration of the box or, equivalently, the force applied to the box (see [148] for the derivation of the model).

In [27], we prove the exact controllability between eigenstates of this system. More precisely, we have the following theorem.

Theorem 8 *For every $n_0, n_f \in \mathbb{N}^*$, there exists $\eta_{n_0}, \eta_{n_f} > 0$ such that, for every (ϕ_0, S_0, D_0), $(\psi_f, S_f, D_f) \in [\mathcal{S} \cap H^7_{(0)}] \times \mathbb{R} \times \mathbb{R}$ with*

$$\|\psi_0 - \varphi_{n_0}\|_{H^7} + |S_0| + |D_0| < \eta_{n_0}, \quad \|\psi_f - \varphi_{n_f}\|_{H^7} + |S_f| + |D_f| < \eta_{n_f},$$

there exists a time $\tau > 0$ and a trajectory (ψ, S, D, u) of the system (3.14) on $[0, \tau]$ such that $(\psi, S, D)(0) = (\psi_0, S_0, D_0)$, $(\psi, S, D)(\tau) = (\psi_f, S_f, D_f)$, $u \in H^1_0((0, \tau), \mathbb{R})$.

Interest : The main interest of this result relies on the globality of the exact controllability result proved, in the context of infinite dimensional bilinear systems.

Technics : The proof relies on local exact controllability results, coupled with a compactness argument. In order to prove the local exact controllability results, in addition to the technics already used in [22], we use power series expansions. Indeed, because of the introduction of the variables S and D, few directions are missed in the controllability of the linearized system. We recover them by using higher order terms (here second order terms are sufficient, but third order terms are sometimes necessary, as, for instance, in [66]).

3.1.5 Other results about bilinear systems

First, let us quote some approximate controllability results for the Schrödinger equation. In [32] Mirrahimi and Beauchard proved the global approximate controllability, in infinite time, for a 1D model, by adapting the approach of [131] where Mirrahimi proved a similar result for equations involving a continuous spectrum. Approximate controllability, in finite time, has been proved for particular models by Boscain and Adami in [1], by using adiabatic theory and intersection of the eigenvalues in the space of controls. Approximate controllability, in finite time, for more general models, have been studied by 3 teams, with different tools : by Boscain, Chambrion, Mason, Sigalotti [54], with geometric control methods ; by Nersesyan [137, 138] with feedback controls and variational methods ; and by Ervedoza and Puel [80], on the trapped ion model, thanks to a simplified model.

Optimal control techniques have also been investigated for Schrödinger equations with a non linearity of Hartee type in [19, 20] by Baudouin, Kavian, Puel and in [77] by Cances, Le Bris, Pilot. An algorithm for the computation of such optimal controls is studied in [21] by Baudouin and Salomon.

Finally, let us quote some references concerning bilinear wave equations. In [112, 111, 110] Khapalov considers nonlinear wave equations with bilinear controls. He proves the global approximate controllability to nonnegative equilibrium states.

3.2 Technics used in this chapter

3.2.1 Linearization principle

The linearization principle is the classical approach to prove the local controllability of a nonlinear control system, in a neighborhood of a reference trajectory :

- first, one proves the controllability of the linearized system around this trajectory,
- then, we get the local controllability of the nonlinear system by applying an inverse mapping theorem (or fixed point theorem) to the end-point map.

3.2.2 Moment method

The moment method may be used in the first part of the linearization principle. It consists in formulating the control problem for a linear system into a moment problem on the control function. In this chapter, we always get a trigonometric moment problem, with complex valued exponentials satisfying a suitable Ingham inequality (see [99]). We refer to [16, 113] for more details about the method, or [31, Appendix B] for the case of trigonometric moment problems studied in this chapter.

3.2.3 Nash-Moser theorem

Sometimes, the results one has concerning the well posedness of the nonlinear system are not sufficient to conclude with the classical inverse mapping theorem, in the linearization principle. For example, in [22], by using only classical tools, the nonlinear Cauchy problem does not seem to be well posed in appropriate spaces (the ones in which the linearized system is controllable) : there is an a priori loss of regularity. It is the reason why I applied the Nash-Moser theorem. Roughly speaking, This theorem requires 3 assumptions :

- families of spaces $(E_a)_{a\geqslant 0}$, $(F_b)_{b\geqslant 0}$, with appropriate smoothing operators,
- a C^2 nonlinear map $\Theta : E_a \to F_a$ with appropriate bounds on $d^2\Theta$,
- a surjective differential $d\Theta(x)$ at every point x of a small neighborhood of 0, with a 'tame' estimate on $d\Theta(x)^{-1}$.

The third assumption is often the more difficult to check. In [22], it consists in proving the controllability of an infinite number of linear systems, with an appropriate bound on the controls used. We refer to [114] by Hörmander and [9] by Alinhac and Gérard for a more detailed description of this method and to [97] by Hamilton for other examples of applications.

3.2.4 Coron's return method

Sometimes, the linearized system around the reference trajectory is not controllable. In this situation, one may try to use Coron's return method, which consists in

- finding another reference trajectory, with better controllability properties (for instance a controllable linearized system),
- proving that one may move the solution of the system from the first reference trajectory to the second one (for instance, in [22], we use quasi-static deformations).

This method has been introduced by Coron in [60] to solve a stabilization problem. The return method has been used successfully to prove the controllability of many systems, see for example [63, 61, 62] by Coron for shallow water and Euler equations, [69] by Coron and

Fursikov for Navier Stokes equation [89] by Fursikov and Imanuvilov for Navier-Stokes and Boussinesq equations, [91, 92, 93] by Glass for Euler and Vlasov Poisson equations, [102] by Horsin for Burgers equation, [154] by Sontag for driftless systems, [56, 55] by Chapouly for Navier-Stokes and Burgers equations, [71] by Coron and Guerrero for Navier-Stokes equation, [106] by Coron, Glass and Wang for hyperbolic systems. We also refer to [65] for a more detailled description of this method.

3.2.5 Power series expansions

This method may be tried when the linearized system around the reference trajectory is not controllable. It consists in using higher order terms in order to recover the directions which are missed by the first order term. In a first step, one proves that one may move the second order term in any missed direction (without moving the first order one). Then, we conclude by applying a fixed point theorem.

This method is classical to study the local controllability of control systems in finite dimension It has also been applied successfully to PDEs : to the Korteweg-de Vries equation, by Coron and Crépeau in [66] and by Cerpa and Crépeau in [53].

In [66], only one direction is missed ; the second order term is not sufficient and the authors used the third order term. In [53], a finite (but arbitrarily large) number of directions are missed ; the authors recover them, thanks to the second order term. Their argument is subtle :

- first, they prove that any missed direction corresponds to a non vanishing quadratic form on the second order term,
- then, they use rotations in the complex plane, in order to generate any missed direction.

For this strategy, it is essential that all the missed directions 'turn' in the complex plane with different speeds. We will see in Section 3.7.3 that it is not always the case.

3.3 Exact controllability of a particle in a variable domain [23] (A4)

3.3.1 Equation and result

In the article [23], we consider a quantum particle in a 1D infinite square potential well, with variable length $l(\tau)$, where

$$\begin{array}{rccc} l: & [0,+\infty) & \to & (0,+\infty) \\ & \tau & \mapsto & l(\tau) \end{array}$$

is a continuous function of the time variable τ. At any time τ, the particle is represented by a wave function $\phi(\tau,z)$, such that, for every τ, $\phi(\tau,z)$ is defined for $z \in (0,l(\tau))$ and satisfies

$$\int_0^{l(\tau)} |\phi(\tau,z)|^2 dz = 1. \tag{3.15}$$

This wave function is solution of the following Schrödinger equation

$$\left\{ \begin{array}{l} i\frac{\partial\phi}{\partial\tau}(\tau,z) = -\frac{\partial^2\phi}{\partial z^2}(\tau,z), \tau \in \mathbb{R}_+^*, z \in (0,l(\tau)), \\ \phi(\tau,0) = \phi(\tau,l(\tau)) = 0, \tau \in \mathbb{R}_+^*. \end{array} \right. \tag{3.16}$$

It is a nonlinear control system in which the state is the wave function ϕ and the control is the real valued function l.

In order to work on a more convenient control system, we perform changes of space variable $z \to x$, time variable $\tau \to t$, wave function $\phi(\tau, z) \to \psi(t, x)$, and control $l \to u$. They lead to the equivalent nonlinear control system

$$\begin{cases} i\frac{\partial \psi}{\partial t}(t,x) = -\frac{\partial^2 \psi}{\partial x^2}(t,x) + [\dot{u}(t) - 4u^2(t)]x^2\psi(t,x), t \in \mathbb{R}_+^*, x \in (0,1), \\ \psi(t,0) = \psi(t,1) = 0, t \in \mathbb{R}_+^* \end{cases} \tag{3.17}$$

in which

- the state variable is the wave function ψ, with $\int_0^1 |\psi(t,x)|^2 dx = 1$ for every t,
- the control is the real valued time depending function u, with $u(0) = u(t_f) = 0$, $\int_0^{t_f} u(s)ds = 0$ where t_f is the final time.

The constraints on the control come from the changes of variable.

We give a sense to the solution of the initial problem (3.16), posed on a variable domain, by using this definition of solution for the new system (3.17) posed on a fixed domain : given a regular function $l : [0, +\infty) \to (0, +\infty)$ (regular enough so that the corresponding function u is C^1).

Let us introduce the operator

$$D(A) := H^2 \cap H_0^1((0,1), \mathbb{C}), \quad A\varphi := -\varphi'',$$

its eigenvalues $\lambda_k := (k\pi)^2, \forall k \in \mathbb{N}^*$, associated eigenvectors $\varphi_k(x) := \sqrt{2}\sin(k\pi x), \forall k \in \mathbb{N}^*$ and the spaces $H^s_{(0)} := D(A^{s/2})$ for every $s > 0$.

In [23], we prove the following result.

Theorem 9 *Let $\epsilon > 0$. For every $n \in \mathbb{N}^*$, there exists $\eta_n > 0$ such that, for every $n_0, n_f \in \mathbb{N}^*$, for every $\psi_0, \psi_f \in \mathbb{S} \cap H^{5+\epsilon}_{(0)}$ with*

$$\|\psi_0 - \varphi_{n_0}\|_{H^{5+\epsilon}} < \eta_{n_0}, \quad \|\psi_f - \varphi_{n_f}\|_{H^{5+\epsilon}} < \eta_{n_f},$$

there exists a time $\mathcal{T}$ and a trajectory (ψ, u) of (3.17) on $[0, \mathcal{T}]$ that satisfies $\psi(0) = \psi_0$, $\psi(\mathcal{T}) = \psi_f$, $u \in H^2 \cap H_0^1((0, \mathcal{T}), \mathbb{R})$ and $\int_0^{\mathcal{T}} u(t)dt = 0$.

Thus, we also have the following corollary.

Theorem 10 *For every $n_0, n_f \in \mathbb{N}^*$, there exists $\mathcal{T} > 0$ and a trajectory (ϕ, l) of (3.16) on $[0, \mathcal{T}]$ such that $l \in C^2([0, \mathcal{T}], \mathbb{R}_+^*)$, $l(0) = l(\mathcal{T}) = 1$, $\phi(0) = \varphi_{n_0}$, $\phi(\mathcal{T}) = \varphi_{n_f}$.*

3.3.2 Interest

An interest of this work consists in transforming a strongly nonlinear control system into a standard bilinear one, which allows to use the technics developed for these last ones. However, the interaction between the control and the state is more non linear than in the previous works [22, 27] (the interaction is of the form $[\dot{u}(t) - 4u^2(t)]x^2\psi(t,x)$ instead of $u(t)\mu(x)\psi(t,x)$) ; this justifies few adaptations.

Finally, a gain in the regularity is done with respect to the previous works [22, 27] : with H_0^1-controls, here, we prove the controllability in H^{5+}, instead of H^7 in [22, 27]. This improvement is allowed by a more careful application of the Nash-Moser theorem.

3.3.3 Technics and remarks

As in [27], the global strategy relies on a compactness argument, that uses the local exact controllability around an infinite number of reference trajectories. These local results are proved with the linearization principle and the Nash-Moser implicit function theorem. For particular trajectories, the linearized system is not controllable and misses a finite number of directions, that are recovered with second order terms.

In [23], we notice that it is sufficient to prove Theorem 9 with $n_f = n_0 + 1$. Then, we prove it only for $n_0 = 1$ and $n_f = 2$, pretending that the general case may be treated as well. Actually, the general case requires few adaptations.

Indeed, with $n_0 = 1$, $n_f = 2$, only one (real) direction is lost on the linearized system around ψ_1 and ψ_2. This is the case around ψ_K as soon as

$$\forall k_1 \in \mathbb{N} \cap [1, K-1], \forall k_2 \in \mathbb{N} \cap [K+1, +\infty), 2K^2 \neq k_1^2 + k_2^2,$$

for example $K \in \{1, 2, 3, 4, 6\}$. But it is not always the case, for example, for $K = 5$, one may take $k_1 = 1$ and $k_2 = 7$. In such a degenerate case, the linearized system around ψ_K losses exactly $(2N + 1)$ directions (in the 2 senses), where $N := \sharp\mathcal{D}$ and

$$\mathcal{D} := \{(k_1, k_2) \in \mathbb{N}^2; 1 \leqslant k_1 < K < k_2 < +\infty, 2K^2 = k_1^2 + k_2^2\}.$$

These directions can be recovered thanks to the second order term, by applying the technics developed in [53]. Indeed,

- the N (complex) directions 'turn' with different speeds,
- on the second order term, they are associated to non vanishing quadratic forms, as proved in the following Lemma.

Lemma 1 *Let $T \geqslant 4/\pi$ and $K \in \mathbb{N}^*$ be such that*

$$\exists k_1, k_2 \in \mathbb{N} \text{ such that } 1 \leqslant k_1 < K < k_2, 2K^2 = k_1^2 + k_2^2. \tag{3.18}$$

We introduce the function

$$h(t, \tau) := \sum_{k=1}^{\infty} b_k^{(1)} e^{i[(\lambda_k - \lambda_{k_1})t + (\lambda_K - \lambda_k)\tau]} + b_k^{(2)} e^{i[(\lambda_{k_2} - \lambda_k)t + (\lambda_k - \lambda_K)\tau]},$$

where

$$b_k^{(j)} := \frac{\langle x^2 \varphi_K, \varphi_k \rangle \langle x^2 \varphi_k, \varphi_{k_j} \rangle}{\langle x^2 \varphi_K, \varphi_{k_j} \rangle} \text{ for } j = 1, 2.$$

The quadratic form

$$Q(v) := \int_0^T v(t) \int_0^t v(\tau) h(t, \tau) d\tau dt$$

is not identically equal to zero on

$$X_T := \left\{ v \in L^2((0, T), \mathbb{R}); \int_0^T v(t) e^{i(\lambda_k - \lambda_K)t} dt = 0, \forall k \in \mathbb{N}^* \right\}.$$

Heuristic of the sufficiency of Lemma 1 : We consider power series expansions

$$\psi = \psi_K + \epsilon\Psi + \epsilon^2\xi + ... \quad u = \epsilon v + \epsilon^2 w + ...$$

where $v, w \in H_0^1((0,T),\mathbb{R})$ and $\int_0^T v(t)dt = \int_0^T w(t)dt = 0$. We have

$$\begin{cases} i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} + \dot{v}x^2\psi_K, \\ \Psi(t,0,) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0, \end{cases}$$

$$\begin{cases} i\frac{\partial\xi}{\partial t} = -\frac{\partial^2\xi}{\partial x^2} + [\dot{w} - 4v^2]\psi_K + \dot{v}x^2\Psi, \\ \xi(t,0,) = \xi(t,1) = 0, \\ \xi(0,x) = 0. \end{cases}$$

Thus,

$$\Psi(T,x) = -i\sum_{k=1}^{\infty}\langle x^2\varphi_K, \varphi_k\rangle \int_0^T \dot{v}(t)e^{i(\lambda_k-\lambda_K)t}dte^{-i\lambda_K T}\varphi_k(x).$$

The assumption (3.18) is equivalent to $\lambda_{k_2} - \lambda_K = -(\lambda_{k_1} - \lambda_K)$, so we have

$$\frac{\langle\Psi(T),\varphi_{k_2}\rangle e^{i\lambda_{k_2}T}}{(-i)\langle x^2\varphi_{k_2},\varphi_K\rangle} = \frac{\overline{\langle\Psi(T),\varphi_{k_1}\rangle} e^{-i\lambda_{k_1}T}}{i\langle x^2\varphi_{k_1},\varphi_K\rangle}, \forall(k_1,k_2) \in \mathcal{D}.$$

Let $(k_1,k_2) \in \mathcal{D}$ and let us try to move in the following direction : $\Psi(T) = 0$ and $\xi(T) = z\varphi_{k_1}$ for any $z \in \{-1,+1,-i,+i\}$. The condition $\Psi(T) = 0$ is equivalent to

$$\int_0^T \dot{v}(t)e^{i(\lambda_k-\lambda_K)t}dt = 0, \forall k \in \mathbb{N}^*$$

because

$$\langle x^2\varphi_K,\varphi_k\rangle = \frac{8(-1)^{k+K}kK}{(k+K)^2(k-K)^2\Pi^2} \neq 0, \forall k \in \mathbb{N}^*.$$

Easy computations show that

$$\frac{\langle\xi(T),\varphi_{k_2}\rangle e^{i\lambda_{k_2}T}}{(-i)\langle x^2\varphi_{k_2},\varphi_K\rangle} - \frac{\overline{\langle\xi(T),\varphi_{k_1}\rangle} e^{-i\lambda_{k_1}T}}{i\langle x^2\varphi_{k_1},\varphi_K\rangle} = iQ(\dot{v}).$$

Thanks to the technics developed in [53], it is sufficient to prove Lemma 1. □

Proof of Lemma 1 : It is sufficient to prove Lemma 1 with $T = 4/\pi$. Let $\omega := \lambda_K - \lambda_{k_1} = \lambda_{k_2} - \lambda_K$. Let us assume that $Q_T(v) = 0, \forall v \in X_T$. Then, for every $v \in X_T$,

$$\nabla Q_T(v) \in \text{Adh}_{L^2(0,T)}\Big(\text{Span}\{e^{\pm i(\lambda_k-\lambda_K)t}; k \in \mathbb{N}^*\}\Big), \tag{3.19}$$

where

$$\nabla Q_T(v) : t \mapsto \int_0^T v(\tau)[h(t,\tau)1_{t\geqslant\tau} + h(\tau,t)1_{\tau>t}]d\tau.$$

Let us compute $\nabla Q_T(v)$ for

$$v(t) := \begin{cases} +1 \text{ if } t \in [0,T/2], \\ -1 \text{ if } t \in (T/2,T]. \end{cases}$$

This function v belongs to X_T because the functions $t \mapsto e^{i(\lambda_k - \lambda_K)t}$ are $T/2$-periodic. For $t \in [0, T/2]$, we have

$$\nabla Q_T(v) : t \mapsto \int_0^t h(t,\tau)d\tau + \int_t^{T/2} h(\tau,t)d\tau - \int_{T/2}^{T} h(\tau,t)d\tau.$$

Let us call $A + B - C$ this decomposition. Easy computations show that

$$\begin{aligned} A = \ & [b_K^{(1)} + b_K^{(2)}]te^{i\omega t} + \left[\sum_{k=1, k\neq K}^{\infty} \frac{b_k^{(1)} - b_k^{(2)}}{i(\lambda_K - \lambda_k)} \right] e^{i\omega t} \\ & - \sum_{k=1,k\neq K}^{\infty} \left(\frac{b_k^{(1)}}{i(\lambda_K - \lambda_k)} e^{i(\lambda_k - \lambda_{k_1})t} + \frac{b_k^{(2)}}{i(\lambda_k - \lambda_K)} e^{i(\lambda_{k_2} - \lambda_k)t} \right), \end{aligned}$$

$$\begin{aligned} B = \ & [b_{k_1}^{(1)} + b_{k_2}^{(2)}] \left(\tfrac{T}{2} - t \right) e^{i\omega t} + \left[\sum_{k=1,k\neq k_1}^{\infty} \frac{b_k^{(1)}}{i(\lambda_k - \lambda_{k_1})} + \sum_{k=1,k\neq k_2}^{\infty} \frac{b_k^{(2)}}{i(\lambda_{k_2} - \lambda_k)} \right] e^{i\omega t} \\ & - \sum_{k=1,k\neq k_1}^{\infty} \frac{b_k^{(1)}}{i(\lambda_k - \lambda_{k_1})} e^{i(\lambda_K - \lambda_k)t} - \sum_{k=1,k\neq k_2}^{\infty} \frac{b_k^{(2)}}{i(\lambda_{k_2} - \lambda_k)} e^{i(\lambda_k - \lambda_K)t}, \end{aligned}$$

$$C = \frac{T}{2}[b_{k_1}^{(1)} + b_{k_2}^{(2)}]e^{i\omega t}.$$

Thus, the coefficient in front of $t \in [0, T/2] \mapsto te^{i\omega t}$ in $\nabla Q_T(v)$ is

$$D := b_K^{(1)} + b_K^{(2)} - b_{k_1}^{(1)} - b_{k_2}^{(2)} = 2\langle x^2\varphi_K, \varphi_K\rangle - \langle x^2\varphi_{k_1}, \varphi_{k_1}\rangle - \langle x^2\varphi_{k_1}, \varphi_{k_1}\rangle$$

Let us prove that $D \neq 0$. Thanks to the explicit expression $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, we get

$$\langle x^2\varphi_j, \varphi_j\rangle = \frac{1}{3} - \frac{1}{2j^2\pi^2}, \forall j \in \mathbb{N}^*.$$

Thus, using the relation $k_1^2 + k_2^2 = 2K^2$, we get

$$D = \frac{1}{2\pi^2}\left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{2}{K^2} \right) = \frac{K^2(k_2^2 + k_1^2) - 2k_1^2k_2^2}{2\pi^2K^2k_1^2k_2^2} = \frac{(k_1^2 - k_2^2)^2}{4\pi^2K^2k_1^2k_2^2} > 0.$$

We deduce from (3.19) that

$$\nabla Q_T(v) \in \mathrm{Adh}_{L^2(0,T/2)}\Big(\mathrm{Span}\{e^{\pm i(\lambda_k - \lambda_K)t}; k \in \mathbb{N}^*\}\Big). \tag{3.20}$$

Thus, the previous computations prove that

$$t \in Y_T := \mathrm{Adh}_{L^2(0,T/2)}\Big(\mathrm{Span}\{e^{i(\lambda_k - \lambda_K - \omega)t}, e^{i(\lambda_K - \lambda_k - \omega)t}, e^{i(\lambda_k - \lambda_{k_1} - \omega)t}, e^{i(\lambda_{k_2} - \lambda_k - \omega)t}; k \in \mathbb{N}^*\}\Big).$$

Then integrations prove that any polynomial belongs to Y_T. Thanks to the Weierstrass theorem, we deduce that $Y_T = L^2(0, T/2)$ which is a contradiction. $\square$

3.4 Exact controllability of a bilinear beam equation [24] (A5)

3.4.1 Equation and result

In the article [24], we consider an homogeneous beam equation, with clamped ends, with an axial load $p(t)$. Let $u(t,x)$ be the vertical displacement of the beam at time t, at the absis x ($w \equiv 0$ at the equilibrium). Then the function u solves the equation

$$\left\{ \begin{array}{l} u_{tt}(t,x) + u_{xxxx}(t,x) + p(t) u_{xx}(t,x) = 0, x \in (0,1), \\ u(t,0) = u(t,1) = u_x(t,0) = u_x(t,1) = 0. \end{array} \right. \tag{3.21}$$

It is a control system in which
- the state is the couple (w, w_t),
- the control is the axial load $p(t)$.

We introduce the operator A defined by

$$D(A) := H^4 \cap H^2_0((0,1),\mathbb{R}), \quad Av := \frac{d^4 v}{dx^4}. \tag{3.22}$$

Let $(\lambda_n)_{n\in\mathbb{N}^*} \subset \mathbb{R}^*_+$ be the increasing sequence of eigenvalues of A and $(\varphi_n)_{n\in\mathbb{N}^*}$ associated orthonormalised eigenvectors. Then, for every $n \in \mathbb{N}^*$, the functions

$$\varphi_n(x)\cos(\sqrt{\lambda_n}t) \text{ and } \varphi_n(x)\sin(\sqrt{\lambda_n}t)$$

are solutions of (3.21) with $p \equiv 0$. For $s > 0$, we introduce the space

$$H^s_{(0)}((0,1),\mathbb{R}) := D(A^{s/4}), \tag{3.23}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}((0,1),\mathbb{C})} := \left(\sum_{k=1}^{\infty} |\lambda_k^{s/4} \langle \varphi, \varphi_k \rangle|^2 \right)^{1/2}. \tag{3.24}$$

The main result of [24] is the following one.

Theorem 11 *Let $T := 8/\pi$, $\epsilon > 0$ and*

$$u^{ref}(t,x) := \varphi_2(x)\sin(\sqrt{\lambda_2}t) + \varphi_3(x)\sin(\sqrt{\lambda_3}t). \tag{3.25}$$

There exists a neighborhood V_0 of $(u^{ref}(0), \dot{u}^{ref}(0))$ and a neighborhood V_T of $(u^{ref}(T), \dot{u}^{ref}(T))$ in $H^{5+\epsilon}_{(0)} \times H^{3+\epsilon}_{(0)}((0,1),\mathbb{R})$, such that, for every $(u_0, \dot{u}_0) \in V_0$, for every $(u_T, \dot{u}_T) \in V_T$, there exists $p \in H^1_0((0,T),\mathbb{R})$ such that, the solution of (3.21) with $(u(0), \dot{u}(0)) = (u_0, \dot{u}_0)$ and control p satisfies $(u(T), \dot{u}(T)) = (u_T, \dot{u}_T)$.

For the system (3.21), Ball, Marsden and Slemrod's theorem provides the following negative result : for every $(u_0, \dot{u}_0) \in H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$, the set of (u, u_t) accessible from $(u_0, \dot{u}_0)$ with controls in $L^r_{loc}([0,\infty),\mathbb{R})$, $r > 1$ has an empty interior in $H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$. Thus, the system is not exactly controllable in $H^2_0((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$ with L^r_{loc}-controls, $r > 1$. However, we prove that it is exactly controllable, in $H^{5+}_{(0)} \times H^{3+}_{(0)}((0,1),\mathbb{R})$, with $H^1_0((0,T),\mathbb{R})$-controls, locally around the reference trajectory (3.25).

3.4.2 Technics, remarks

The strategy used to prove this result consists in adapting to the beam equation the technics that were already successful for the Schrödinger equation : linearization principle and Nash-Moser theorem. An interest of this result is that this beam equation is one of the historical system on which Ball, Marsden and Slemrod applied their general result (see Subsection 3.1.3).

The result of Section 3.6 should convince the reader that this proof may be simplified, and the controllability result may be optimized : the exact local controllability should hold in $H^3_{(0)} \times H^1_{(0)}((0,1),\mathbb{R})$, with L^2-controls, in any positive time $T > 0$.

Theorem 11 guarantees the local exact controllability in time $T = 8/\pi$. This choice is technical and related to the use of the Nash-Moser theorem. As for the Schrödinger equation (in the favorable case studied in the next section), no minimal time should be required for the controllability : the local exact controllability should hold in any positive time $T > 0$. Again, the technics presented in Section 3.6 should allow to prove this point.

3.5 Exact controllability of a bilinear wave equation [25] (A13)

3.5.1 Equation and result

In this article, we consider a linear wave equation with bilinear control

$$\left\{ \begin{array}{l} w_{tt}(t,x) = w_{xx}(t,x) + u(t)\mu(x)w(t,x), x \in (0,1), \\ w_x(t,0) = w_x(t,1) = 0, \end{array} \right. \tag{3.26}$$

in which the state is the couple (w, w_t) and the control is $u : [0,T] \to \mathbb{R}$. It represents a vibrating string with axial load $u(t)\mu(x)$.

Let us introduce some conventions and notations. The operator A is defined by

$$D(A) := \{\varphi \in H^2(0,1); \varphi'(0) = \varphi'(1) = 0\}, \quad A\varphi := -\varphi''. \tag{3.27}$$

Its eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ and eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$ are

$$\begin{array}{ll} \lambda_0 := 0, & \varphi_0(x) := 1, \\ \lambda_k := (k\pi)^2, & \varphi_k(x) := \sqrt{2}\cos(k\pi x), \forall k \in \mathbb{N}^*. \end{array} \tag{3.28}$$

We define the spaces

$$H^s_{(0)}(0,1) := D(A^{s/2}), \forall s > 0 \tag{3.29}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}} := \left(\sum_{k=0}^{\infty} |k_*^s \langle \varphi, \varphi_k \rangle|^2 \right)^{1/2},$$

where $k_* := \max\{k, 1\}, \forall k \in \mathbb{N}$ and $\langle .,. \rangle$ is the $L^2(0,1)$-scalar product.

The goal of the article [25] is to prove that, under generic assumptions on μ, the system (3.26) is locally controllable around the reference trajectory $(w(t,x)=1, u(t)=0)$, if and only if $T>2$. The restriction $T>2$ is not surprising because this wave equation has a propagation speed equal to 1, but, in this article, a particular attention is given to the case $T\leqslant 2$. Precisely, we prove the following results.

- When $T>2$, the system (3.26) is locally controllable in $H^3_{(0)}\times H^2_{(0)}(0,1)$ with $L^2(0,T)$-controls.
- When $T=2$, the system (3.26) is not locally controllable in $H^3_{(0)}\times H^2_{(0)}(0,1)$ with $L^2(0,T)$-controls because the reachable set is (locally) a non flat submanifold of $H^3_{(0)}\times H^2_{(0)}(0,1)$ with codimension one. However, the system (3.26) is locally controllable up to codimension one : one can control the couple $(w-\int_0^1 w(x)dx, \partial w/\partial t)$. Moreover, for any reachable (local) target, there exists a unique (small) control allowing to reach this target.
- When $T<2$, the system (3.26) is strongly not controllable : the reachable set, with $L^2(0,T)$-controls, is (locally) contained in a non flat submanifold of $H^3_{(0)}\times H^2_{(0)}(0,1)$ with infinite codimension.

The main result of the article [25] is the following one.

Theorem 12 *Let $\mu\in H^2(0,1)$. We assume*

$$\exists c>0 \text{ such that } \frac{c}{k_*^2}\leqslant |\langle \mu,\varphi_k\rangle|, \forall k\in\mathbb{N}. \tag{3.30}$$

(1) *Let $T>2$. There exists $\delta>0$ and a C^1-map*

$$\begin{array}{rccl}\Gamma_T: & \mathcal{V}_T & \to & L^2(0,T)\\ & (w_f,\dot{w}_f) & \mapsto & \Gamma_T(w_f,\dot{w}_f)\end{array}$$

where

$$\mathcal{V}_T:=\{(w_f,\dot{w}_f)\in H^3_{(0)}\times H^2_{(0)}(0,1); \|w_f-1\|_{H^3_{(0)}}+\|\dot{w}_f\|_{H^2_{(0)}}<\delta\},$$

such that, $\Gamma_T(1,0)=0$ and for every $(w_f,\dot{w}_f)\in\mathcal{V}_T$, the solution of (3.26) with initial condition

$$\left(w,\frac{\partial w}{\partial t}\right)(0,x)=(1,0), \forall x\in(0,1), \tag{3.31}$$

and control $u=\Gamma_T(w_f,\dot{w}_f)$ satisfies $(w,\frac{\partial w}{\partial t})(T)=(w_f,\dot{w}_f)$.

(2) *Let $T=2$. There exists $\delta,r>0$ and a C^1 map*

$$\begin{array}{rccl}\Gamma_T: & \mathcal{V}_T & \to & B_r[L^2(0,T)]\\ & (\tilde{w}_f,\dot{w}_f) & \mapsto & \Gamma_T(\tilde{w}_f,\dot{w}_f)\end{array}$$

where

$$\mathcal{V}_T:=\{(\tilde{w}_f,\dot{w}_f)\in H^3_{(0)}\times H^2_{(0)}(0,1); \int_0^1 \tilde{w}_f(x)dx=0, \|\tilde{w}_f\|_{H^3_{(0)}}+\|\dot{w}_f\|_{H^2_{(0)}}<\delta\},$$

$$B_r[L^2(0,T)]:=\{u\in L^2((0,T),\mathbb{R}); \|u\|_{L^2}<r\},$$

such that, $\Gamma_T(0,0)=0$ and for every $(\tilde{w}_f,\dot{w}_f)\in\mathcal{V}_T$, $u\in B_r[L^2(0,T)]$, the solution of (3.26), (3.31) satisfies

$$w(T)-\int_0^1 w(T,x)dx=\tilde{w}_f \text{ and } \frac{\partial w}{\partial t}(T)=\dot{w}_f,$$

if and only if $u=\Gamma_T(\tilde{w}_f,\dot{w}_f)$.

The reachable set from (3.31) is, locally, a C^1-submanifold with codimension one. More precisely, there exists $r'>0$ and a locally surjective non linear C^1-map $G_T: H^3_{(0)}\times H^2_{(0)}(0,1)\to\mathbb{R}$ such that, for every $u\in B_{r'}[L^2(0,T)]$, the solution of (3.26), (3.31) satisfies

$$G_T\left[\left(w,\frac{\partial w}{\partial t}\right)(T)\right]=0.$$

(3) We assume

$$\frac{(\mu^2)'(1)\pm(\mu^2)'(0)}{\mu'(1)\pm\mu'(0)}\neq\frac{\int_0^1\mu(x)^2dx}{\int_0^1\mu(x)dx}. \tag{3.32}$$

Let $T<2$. The reachable set from (3.31) is, locally, contained in a C^1-submanifold of $H^3_{(0)}\times H^2_{(0)}(0,1)$, with infinite codimension, that does not coincide with its tangent space at $(1,0)$. More precisely, there exists $r>0$, a strict vector subspace R_T of $H^3_{(0)}\times H^2_{(0)}(0,1)$ with infinite dimension and a locally surjective C^1 map

$$G_T: H^3_{(0)}\times H^2_{(0)}(0,1)\to R_T$$

such that, for every $u\in B_r[L^2(0,T)]$, the solution of (3.26), (3.31) satisfies

$$G_T\left[\left(w,\frac{\partial w}{\partial t}\right)(T)\right]=0.$$

Remark 1 *Notice that, when (3.30) holds, then $\int_0^1\mu=\langle\mu,\varphi_0\rangle\neq0$ and $\mu'(1)\pm\mu'(0)\neq0$. Indeed, we have*

$$\langle\mu,\varphi_k\rangle=\frac{\sqrt{2}}{(k\pi)^2}\Big((-1)^k\mu'(1)-\mu'(0)\Big)-\frac{\sqrt{2}}{(k\pi)^2}\int_0^1\mu''(x)\cos(k\pi x)dx. \tag{3.33}$$

This remark gives a sense to each term in (3.32).

Remark 2 *The assumptions (3.30) and (3.32) hold simultaneously, for example, with $\mu(x)=x^2$, because*

$$\begin{aligned}&\langle x^2,\varphi_0\rangle=\textstyle\int_0^1 x^2dx=\frac{1}{3},\\&\langle x^2,\varphi_k\rangle=\textstyle\int_0^1 x^2\sqrt{2}\cos(k\pi x)dx=\frac{(-1)^k2\sqrt{2}}{(k\pi)^2},\forall k\in\mathbb{N}^*,\end{aligned} \tag{3.34}$$

$$\frac{(\mu^2)'(1)\pm(\mu^2)'(0)}{\mu'(1)\pm\mu'(0)}=2, \text{ and } \frac{\int_0^1\mu(x)^2dx}{\int_0^1\mu(x)dx}=\frac{3}{5}.$$

But (3.30) and (3.32) are not always satisfied. For example, (3.30) does not hold when $\langle\mu,\varphi_k\rangle=0$ for some $k\in\mathbb{N}$, or when μ has a symmetry with respect to $x=1/2$. However, the assumptions (3.30) and (3.32) are generic in $H^2(0,1)$ (see [25, Appendix] for a proof), thus, the previous theorem is very general.

Remark 3 *In Theorem 12, the spaces are optimal. Indeed, for every control $u \in L^2(0,T)$, there exists a unique solution of (3.26), (3.31) and it satisfies*

$$\left(w, \frac{\partial w}{\partial t}\right)(T) \in H^3_{(0)} \times H^2_{(0)}(0,1).$$

3.5.2 Technics and interest

The first interest of this work consists in proving exact controllability for an infinite dimensional bilinear system, without applying the Nash-Moser theorem. Indeed, a smoothing effect is easy to emphasize (the proof relies on integrations by parts and Bessel Parseval equality) and allows to conclude thanks to the inverse mapping theorem.

A second interest of this work consists in proving strong negative results for bilinear systems : the non controllability result, for $T < 2$ is much stronger than Ball, Marsden and Slemrod's one. Indeed, it prevents the reachable set from being (locally) a whole functional space strictly smoother than $H^3_{(0)} \times H^2_{(0)}(0,1)$. This type of negative result had already been proved for the Bloch equation [29].

Another interest of this work is to improve the understanding of the controllability of 2D bilinear Schrödinger equations. Indeed, the equation (3.26) may be seen as a toy model, from the spectral point of view, for the equation

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\Delta\psi - u(t)\mu(x)\psi, x \in \Omega, \\ \psi(t,x) = 0, x \in \partial\Omega, \end{cases} \tag{3.35}$$

where Ω is a bounded regular open subset of $\mathbb{R}^2$. The behavior of the eigenvalues of the Dirichlet-Laplacian operator is not well understood on 2D bounded open domains (in particular, no gap condition is known, in general). Thus, the exact controllability of this equation is an ambitious open problem. For the wave equation (3.26), the spectrum of the operator satisfies the same Weyl formula than the 2D-Dirichlet-Laplacian operator, but it has more structure (in particular, there is a uniform gap), thus, the analysis is easier.

Concerning the technics, the proof of the three points relies on the linearization principle and the inverse mapping theorem. When $T < 2$, these technics allow to prove that the reachable set is a submanifold of $H^3_{(0)} \times H^2_{(0)}(0,1)$ with infinite codimension. Then, by considering the second order term, we check that this submanifold does not coincide with its tangent space at the point $(1,0)$, thus the submanifold is 'not flat'.

3.6 A simpler proof [31] (A12)

3.6.1 Equation and result

In the article [31], in a first step, we focus on the 1D linear Schrödinger equation

$$\begin{cases} i\frac{\partial \psi}{\partial t}(t,x) = -\frac{\partial^2 \psi}{\partial x^2}(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0, \end{cases} \tag{3.36}$$

where $\mu \in H^3((0,1),\mathbb{R})$. It represents a quantum particle, in a 1D infinite square potential well, in an electric field. It is a control system in which

- the state is ψ, with $\|\psi(t)\|_{L^2(0,1)} = 1$, $\forall t \in (0,T)$,

– the control is the real valued function u.
Let us introduce the operator A, defined by

$$D(A) := H^2 \cap H_0^1((0,1),\mathbb{C}), \quad A\varphi := -\frac{d^2\varphi}{dx^2}, \tag{3.37}$$

its eigenvalues and eigenvectors are

$$\lambda_k := (k\pi)^2, \varphi_k(x) := \sqrt{2}\sin(k\pi x), \forall k \in \mathbb{N}^*. \tag{3.38}$$

Then,

$$\psi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \forall k \in \mathbb{N}^*$$

is a solution of (3.36) with $u \equiv 0$ called eigenstate, or ground state, when $k = 1$. We define the spaces

$$H^s_{(0)}((0,1),\mathbb{C}) := D(A^{s/2}), \forall s > 0 \tag{3.39}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}} := \left(\sum_{k=1}^{\infty} |k^s \langle \varphi, \varphi_k \rangle|^2 \right)^{1/2}.$$

We denote by $\langle .,. \rangle$ the $L^2((0,1),\mathbb{C})$ scalar product

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx$$

and by $\mathcal{S}$ the unit $L^2((0,1),\mathbb{C})$-sphere. The first result of the article [31] is the following one.

Theorem 13 *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\exists c > 0 \text{ such that } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k \rangle|, \forall k \in \mathbb{N}^*. \tag{3.40}$$

There exists $\delta > 0$ and a C^1 map

$$\Gamma: \ \mathcal{V}_T \ \rightarrow \ L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^3_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (3.36) with initial condition

$$\psi(0) = \varphi_1 \tag{3.41}$$

and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

Remark 4 *The assumption (3.40) holds, for example, with $\mu(x) = x^2$, because*

$$\langle x^2\varphi_1, \varphi_k \rangle = \int_0^1 2x^2 \sin(k\pi x)\sin(\pi x)dx = \begin{cases} \frac{(-1)^{k+1}8k}{\pi^2(k^2-1)^2} \text{ if } k \geqslant 2, \\ \frac{-3+2\pi^2}{6\pi^2} \text{ if } k = 1. \end{cases} \tag{3.42}$$

But it does not hold when $\langle \mu\varphi_1, \varphi_k \rangle = 0$, for some $k \in \mathbb{N}^*$, or when μ has a symmetry with respect to $x = 1/2$. However, the assumption (3.40) holds generically with respect to $\mu \in H^3((0,1),\mathbb{R})$ because

$$\langle \mu\varphi_1, \varphi_k \rangle = \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu\varphi_1)'''(x)\cos(k\pi x)dx, \forall k \in \mathbb{N}^*. \tag{3.43}$$

(see [31, Appendix] for a proof). Thus, Theorem 1 is very general.

3.6.2 Technics and interest

The local exact controllability of 1D Schrödinger equations, with bilinear control, had already been investigated in [22, 23, 27]. In these articles, the most difficult part of the proof is the application of the inverse mapping theorem. Indeed, because of an a priori loss of regularity, we were led to apply the Nash-Moser implicit function theorem, instead of the classical inverse mapping theorem. The Nash-Moser theorem requires, in particular, the controllability of an infinite number of linearized systems, and tame estimates on the corresponding controls. These two points are difficult to prove and lead to long technical developments in [22, 23, 27].

In this article, we propose a simpler proof, that uses only the classical inverse mapping theorem (needing the controllability of only one linearized system), because we emphasize a hidden regularizing effect, proved with elementary tools.

Therefore, the controllability result of Theorem 13 enters the classical framework of local controllability results for nonlinear systems, proved with fixed point arguments (see, for instance, [146] by Rosier, [53] by Cerpa and Crépeau, [149] by Russell and Zhang, [169] by Zhang, [170] by Zuazua ; of course, this list is not exhaustive).

Moreover, the result is optimal concerning the functional spaces and the time of control. This is another improvement with respect to the previous references.

Finally, let us emphasize that the local exact controllability result of the article [31] **(A12)** and the global approximate controllability of [137, 138] can be put together in order to get the global exact controllability of 1D models (see [138]).

3.6.3 Additional results

The proof we developed for Theorem 13 is quite robust, thus we could apply it to other situations : other linear PDEs and also nonlinear PDEs, that are presented in the next subsections. This shows that the strategy proposed in this article works for a wide range of bilinear systems.

Generalization to higher regularities

The first situation is the analogue result of Theorem 13, but with higher regularities : we prove the local exact controllability of (3.36) in smoother spaces and with smoother controls. Namely, we prove the following result.

Theorem 14 *Let $T > 0$ and $\mu \in H^5((0,1),\mathbb{R})$ be such that (3.40) holds. There exists $\delta > 0$ and a C^1 map*

$$\begin{array}{rccl} \Gamma: & \mathcal{V}_T & \to & H^1_0((0,T),\mathbb{R}) \\ & \psi_f & \mapsto & \Gamma(\psi_f) \end{array}$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^5_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^5} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (3.36), (3.41) with control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

Of course, the strategy may be used to go further on and prove the local exact controllability of (3.36) around the ground state

– in $H^7_{(0)}(0,1)$ with controls in $H^2_0((0,T),\mathbb{R})$,

– in $H^9_{(0)}(0,1)$ with controls in $H^3_0((0,T),\mathbb{R})$, etc.

On the 3D ball with radial data

The second situation is the analogue result of Theorem 13, but for the Schrödinger equation posed on the three dimensional unit ball B^3 for radial data. In polar coordinates, the Laplacian for radial data can be written

$$\Delta u(r) = \partial_r^2 u(r) + \frac{2}{r}\partial_r u(r).$$

In particular, we have $\Delta\left(\frac{g(r)}{r}\right) = \frac{\partial_r^2 u(r)}{r}$. The eigenfunctions of the Dirichlet operator $A = -\Delta$ with domain $D(A) := H^2_{radial} \cap H^1_0(B^3)$ are $\varphi_k = \frac{sin(k\pi r)}{r\sqrt{2\pi}}$ with eigenvalues $\lambda_k = (k\pi)^2$. Thus, we study the Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,r) = -\Delta\psi(t,r) - u(t)\mu(r)\psi(t,r), r \in (0,1), \\ \psi(t,1) = 0. \end{cases} \tag{3.44}$$

The result we obtain is very similar to Theorem 13.

Theorem 15 *Let $T > 0$ and $\mu \in H^3(B^3,\mathbb{R})$ radial be such that*

$$\exists c > 0 \text{ such that } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle|, \forall k \in \mathbb{N}^*. \tag{3.45}$$

There exists $\delta > 0$ and a C^1 map

$$\Gamma : \mathcal{V}_T \rightarrow L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^3_{(0),rad}(B^3,\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (3.44) with initial condition

$$\psi(0) = \varphi_1 \tag{3.46}$$

and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

The analysis is very close to the 1D case since for this particular data, the Laplacian behaves as in dimension 1. Note that this simpler situation has also been used by Anton for proving global existence for the nonlinear Schrödinger equation [15].

Nonlinear Schrödinger equations

The third situation concerns nonlinear Schrödinger equations. More precisely we study the following nonlinear Schrödinger equation with Neumann boundary conditions

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) + |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \frac{\partial\psi}{\partial x}(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0. \end{cases} \tag{3.47}$$

It is a nonlinear control system where

– the state is ψ, with $\|\psi(t)\|_{L^2(0,1)} = 1, \forall t \in [0,T]$,

– the control is the real valued function $u : [0,T] \to \mathbb{R}$.
We study its local controllability around the reference trajectory

$$(\psi_{ref}(t,x) := e^{-it}, u_{ref}(t) = 0).$$

More precisely, we prove the following result.

Theorem 16 *Let $T > 0$ and $\mu \in H^2(0,1)$ be such that*

$$\exists c > 0 \text{ such that } \Big| \int_0^1 \mu(x) \cos(k\pi x) dx \Big| \geqslant \frac{c}{\max\{1,k\}^2}, \forall k \in \mathbb{N}. \tag{3.48}$$

There exists $\eta > 0$ and a C^1-map

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^2(0,1); \psi_f'(0) = \psi_f'(1) = 0 \text{ and } \|\psi_f - e^{-iT}\|_{H^2} < \eta\}$$

such that, for every $\psi_f \in \mathcal{V}_T$, the solution of (3.47) with initial condition

$$\psi(0,x) = 1, \forall x \in (0,1) \tag{3.49}$$

and control $u := \Gamma(\psi_f)$ is defined on $[0,T]$ and satisfies $\psi(T) = \psi_f$.

Remark 5 *The assumption (3.48) holds generically in $H^2(0,1)$. Indeed, integrations by parts give*

$$\int_0^1 \mu(x)\cos(k\pi x)dx = \frac{1}{(k\pi)^2}\left((-1)^{k+1}\mu'(1) + \mu'(0) + \int_0^1 \mu''(x)\cos(k\pi x)dx\right), \forall k \in \mathbb{N}^*.$$

Other versions of this result, with higher regularities may be proved : the system is exactly controllable, locally around the reference trajectory
– in $H^4(0,1)$ with controls in $H^1_0(0,T)$,
– in $H^6(0,1)$ with controls in $H^2_0(0,T)$, etc.
Focusing nonlinearities may also be considered.

Nonlinear wave equations

The third situation concerns nonlinear wave equations. More precisely we study the following wave equation with Neumann boundary conditions

$$\begin{cases} w_{tt} = w_{xx} + f(w, w_t) + u(t)\mu(x)(w + w_t), x \in (0,1), t \in (0,T), \\ w_x(t,0) = w_x(t,1) = 0, \end{cases} \tag{3.50}$$

where f is an appropriate nonlinearity, that satisfies, in particular, $f(1,0) = 0$. It is a nonlinear control system where
– the state is (w, w_t),
– the control is the real valued function $u : [0,T] \to \mathbb{R}$.
We study its exact controllability, locally around the reference trajectory

$$(w_{ref}(t,x) = 1, u_{ref}(t) = 0).$$

More precisely, we prove the following result.

Theorem 17 *Let $T > 2$, $\mu \in H^2((0,1),\mathbb{R})$ be such that (3.48) holds and $f \in C^3(\mathbb{R}^2,\mathbb{R})$ be such that $f(1,0) = 0$ and $\nabla f(1,0) = 0$. There exists $\eta > 0$ and a C^1-map*

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{(w_f, \dot{w}_f) \in H^3 \times H^2((0,1),\mathbb{R}); \quad w_f'(0) = w_f'(1) = \dot{w}_f'(0) = \dot{w}_f'(1) = 0 \text{ and } \|w_f - 1\|_{H^3} + \|\dot{w}_f\|_{H^2} < \eta\}$$

such that $\Gamma(1,0) = 0$ and for every $(w_f, \dot{w}_f) \in \mathcal{V}_T$, the solution of (3.50) with initial condition

$$(w, w_t)(0,x) = (1,0), \forall x \in (0,1) \tag{3.51}$$

and control $u := \Gamma(w_f, \dot{w}_f)$ is defined on $[0,T]$ and satisfies $(w, w_t)(T) = (w_f, \dot{w}_f)$.

Other versions of this result, with higher regularities may be proved : the system is exactly controllable, locally around the reference trajectory
– in $H^4 \times H^3(0,1)$ with controls in $H_0^1(0,T)$,
– in $H^5 \times H^4(0,1)$ with controls in $H_0^2(0,T)$, etc.

3.7 Conclusion, open problems, perspectives

3.7.1 Schrödinger equations on variable domains

The first open problem of this Chapter concerns the adaptation of [23] **(A4)** to the multidimensional case. Let us consider the Schrödinger equation posed on a variable domain of $\mathbb{R}^N$,

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\Delta\psi, x \in (I + u(t))(\Omega), \\ \psi(t,x) = 0, x \in \partial[(I + u(t))(\Omega)], \end{cases}$$

where

$$\begin{array}{rccc} u : & \mathbb{R} \times \mathbb{R}^N & \to & \mathbb{R}^N \\ & (t,x) & \mapsto & u(t,x) \end{array}$$

defines a small perturbation of Ω, for every $t \in [0,T]$ and satisfies $u(0,.) = u(T,.) = 0$. It is a control system in which the state is the wave function ψ and the control is the deformation u. Let us compute formally the linearized system around the trajectory $(\psi = \psi_1, u = 0)$, where $\psi_1(t,x) = \varphi_1(x)e^{-i\lambda_1 t}$, λ_1 is the smallest eigenvalue of the Dirichlet-Laplacian operator on Ω and φ_1 is the associated eigenvector. From the first order expansion

$$\psi = \psi_1 + \epsilon\Psi + o(\epsilon), \quad u(t,x) = \epsilon v(t,x) + o(\epsilon),$$

we deduce

$$0 = \psi(t, x + \epsilon u(t,x)) = \epsilon\Big[\Psi(t,x) + \nabla\psi_1(t,x).v(t,x)\Big] + o(\epsilon).$$

Thus the linearized system is

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = -\Delta\Psi, x \in \Omega, \\ \Psi(t,x) = -\frac{\partial \psi_1}{\partial \nu}(t,x)v(t,x).\nu(x), \end{cases}$$

where ν is the exterior normal vector to the boundary. Moreover, $\frac{\partial \psi_1}{\partial \nu}(t,x) \neq 0$, $\forall x \in \partial\Omega$, so the controllability of the linearized system reduces to the controllability of the linear Schrödinger equation with boundary controls. This problem has already been solved by Lebeau [122], in the L^2-framework. Therefore, several problems may be stated :

- in which spaces the nonlinear system is well posed ?
- is the end point map C^1 between these spaces ?
- is Lebeau's result sufficient to apply the inverse mapping theorem ? Or is it necessary to prove the observability inequality in different spaces ?

3.7.2 Exact controllability of the bilinear Schrödinger equation on multidimensional domains

As emphasized in Subsection 3.5.2, the wave equation (3.26) is a toy model for 2D Schrödinger equations, with bilinear controls (3.35). For such systems, the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ of the Dirichlet-Laplacian operator satisfy the Weyl formula

$$\exists d > 0, \alpha \in (0, n/2) \text{ such that } \mathrm{Card}\{k \in \mathbb{N}; \lambda_k \in [0,t]\} = dt + O(t^\alpha) \text{ when } t \to +\infty.$$

We conjecture that, under generic assumptions on μ,

- for every $T > 2\pi/d$, the system (3.35) is locally exactly controllable around the ground state (or any eigenstate) in some function space (to be defined),
- for every $T < 2\pi/d$, the system (3.35) is not locally exactly controllable around the ground state : the reachable set is contained in a non flat submanifold of some functional space (to be defined), with infinite codimension.

Similarly, for 3D Schrödinger equations with bilinear control (i.e. equation (3.35) with Ω a bounded open subset of $\mathbb{R}^3$), we conjecture that, for every $T > 0$, the reachable set is a non flat submanifold of some functional space, with infinite codimension.

Let us present a possible strategy to approach the 2D conjecture. This strategy consists in proving a partial positive result in large time.

In order to simplify, we consider the system

$$\begin{cases} i\partial_t\psi(t,x,y) = -\Delta\psi(t,x,y) - u(t)\mu(x,y)\psi(t,x,y), (x,y) \in \Omega, \\ \partial_\nu\psi(t,x,y) = 0, (x,y) \in \partial\Omega, \end{cases} \tag{3.52}$$

where $\Omega = (0,\pi)^2$ around the reference trajectory

$$(\psi_{ref}(t,x,y) = 1, u_{ref}(t) = 0).$$

Let

$$\begin{array}{ll} \varphi_{0,0}(x,y) := \frac{1}{\pi}, & \lambda_{0,0} := 0, \\ \varphi_{k,l}(x,y) = \frac{2}{\pi}\cos(kx)\cos(ly), & \lambda_{k,l} = k^2 + l^2, \forall (k,l) \in \mathbb{N}^2, \end{array}$$

be the eigenvectors and eigenvalues of the Neuman-Laplacian on Ω. For dipolar moments μ of the form

$$\mu(x,y) = \mu_1(x)\mu_2(y)$$

where $\mu_1, \mu_2 : (0,\pi) \to \mathbb{R}$, the following asymptotic behavior holds generically

$$\exists c > 0 \text{ tel que } |\langle \mu, \varphi_{k,l}\rangle| \geqslant \frac{c}{k_*^2 l_*^2}, \forall k,l \in \mathbb{N},$$

where $k_* := \max\{1,k\}$, $\forall k \in \mathbb{N}$. Notice that the assumption $\mu_1 \neq \mu_2$ is necessary to get the controllability, otherwise, any solution associated to $\psi(0,x,y) = \varphi_{0,0}(x,y)$ satisfies $\psi(t,x,y) = \psi(t,y,x)$ for every $t > 0$.

Let us consider the linearized system around the ground state

$$\begin{cases} i\partial_t \Psi(t,x,y) = -\Delta\Psi(t,x,y) - v(t)\mu(x,y)\psi_{0,0}(t,x,y), (x,y) \in \Omega, \\ \partial_\nu \Psi(t,x,y) = 0, (x,y) \in \partial\Omega, \\ \Psi(0,x,y) = 0 \end{cases} \tag{3.53}$$

and let us introduce the following spaces

$$\mathcal{V} := \left\{ \Psi \in L^2(\Omega); \frac{\langle \Psi, \varphi_{k,l} \rangle e^{i\lambda_{k,l}T}}{\langle \mu, \varphi_{k,l} \rangle} = \frac{\langle \Psi, \varphi_{K,L} \rangle e^{i\lambda_{K,L}T}}{\langle \mu, \varphi_{K,L} \rangle}, \forall (k,l), (K,L) \in \mathbb{N}^2 / \lambda_{k,l} = \lambda_{K,L} \right\}$$

$$\mathcal{H} := \left\{ \xi \in H^2 \cap H^1_0(\Omega); \sum_{k,l=0}^{+\infty} |k_*^2 l_*^2 \langle \xi, \varphi_{k,l} \rangle|^2 < +\infty \right\},$$

$$\mathcal{K}(0,T) := \left\{ v \in L^2((0,T),\mathbb{R}); \sum_{k,l=1}^{+\infty} \left| \int_0^T v(t) e^{-i\lambda_{k,l}t} dt \right|^2 < +\infty \right\}.$$

We have the following result.

Proposition 1 *Let $T > 2\pi$. For every $\Psi_f \in \mathcal{H} \cap \mathcal{V}$, there exists $v \in \mathcal{K}(0,T)$ such that the solution of (3.53) satisfies $\Psi(T) = \Psi_f$.*

Remark 6 *The solution of (3.53) may be expressed explicitly in the following way*

$$\psi(T,x,y) = \sum_{k,l=1}^{\infty} i \langle \mu, \varphi_{k,l} \rangle \int_0^T v(t) e^{i\lambda_{k,l}t} dt e^{-i\lambda_{k,l}T} \varphi_{k,l}(x,y).$$

In particular, for every $v \in L^2((0,T),\mathbb{R})$ then $\Psi(T) \in \mathcal{V}$, thus, one may not control more than $P\Psi(T)$ (the projection P is defined below). Moreover, for the target $\Psi_f \in \mathcal{H}$, if $v \in L^2((0,T),\mathbb{R})$ allows to reach Ψ_f then, necessarily $v \in \mathcal{K}(0,T)$.

Now, let us present our strategy to approach the 2D conjecture.

Let $\mathcal{F}$ be a family of couples $(k,l) \in \mathbb{N}^2$ such that

- $\lambda_{k,l} \neq \lambda_{K,L}$ for every $(k,l) \neq (K,L) \in \mathcal{F}$,
- $\{\lambda_{k,l}; (k,l) \in \mathcal{F}\} = \{\lambda_{k,l}; (k,l) \in \mathbb{N}^2\}$.

Let us introduce the space

$$V := \mathrm{Adh}_{L^2(\Omega)} \Big(\mathrm{Span}\{\varphi_{k,l}; (k,l) \in \mathcal{F}\} \Big)$$

and the orthogonal projection $P : L^2(\Omega) \to V$.

The strategy we propose is the following one. One may try to prove that the system (3.52) is locally controllable, in a submanifold of $\mathcal{H}$ having V as tangent space at the point $\psi_{0,0}(T)$. This local controllability would hold in any time $T > 2\pi$, locally around ψ_{ref}.

In 1D, the proof of the local exact controllability is in two steps : in a first step, one proves the controllability of the linearized system, in a second step, one proves the well-posedness of the nonlinear system in the same spaces. In 2D, the controllability of the linearized system is linked to difficult problems from harmonic analysis. The strategy proposed above allows to avoid these harmonic analysis problems and to focus only on the nonlinear part of the proof.

However, notice that many problems still appear. For example, on may prove that when $v \in \mathcal{K}(0,T)$, then, the solution of the linearized system does not necessary live in $\mathcal{H}$ on the whole interval $[0,T]$. Thus, the fixed point approach used in 1D (for the well posedness of the nonlinear Cauchy problem), does not work for this 2D problem : the solution of the nonlinear system associated to a control $v \in \mathcal{K}(0,T)$ may satisfy $\psi(T) \in \mathcal{H}$, but it should not live in $\mathcal{H}$ on the whole interval $[0,T]$.

3.7.3 Simultaneous controllability of N Schrödinger equations

Let us consider N independent quantum particles in a 1D infinite square potential well, subjected to the same electric field u,

$$\begin{cases} i\frac{\partial \phi_j}{\partial t}(t,x) = -\frac{\partial^2 \phi_j}{\partial x^2}(t,x) - \epsilon_j u(t)\mu(x)\phi_j(t,x), x \in (0,1), j \in \{1,...,N\}, \\ \phi_j(t,0) = \phi_j(t,1) = 0, j \in \{1,...,N\}, \end{cases} \tag{3.54}$$

where $\epsilon_1,...,\epsilon_N \in \mathbb{R}^*$ are N different real numbers. It is a control system in which
- the state is $(\phi_1,...,\phi_N)$ with $\|\phi_j(t)\|_{L^2} = 1, \forall t \in \mathbb{R}_+, \forall j \in \{1,...,N\}$,
- the control is the real valued function u.

We investigate the local exact controllability around the reference trajectory consisting of the N first eigenvectors

$$\Big((\phi_1^{ref},...,\phi_N^{ref})(t,x) := (\psi_1,...,\psi_N)(t,x), u^{ref}(t) = 0 \Big).$$

This question is addressed in [162] for ODE models. It is related to the control of molecular dynamics [162] and also to the problem of quantum gate generation in information theory [152].

Let us consider the linearized system around this reference trajectory.

$$\begin{cases} i\frac{\partial \Phi_j}{\partial t}(t,x) = -\frac{\partial^2 \Phi_j}{\partial x^2}(t,x) - \epsilon_j v(t)\mu(x)\psi_j(t,x), x \in (0,1), j \in \{1,...,N\}, \\ \Phi_j(t,0) = \Phi_j(t,1) = 0, j \in \{1,...,N\}, \\ \Phi_j(0,x) = 0, j \in \{1,...,N\}. \end{cases} \tag{3.55}$$

The solution is

$$\Phi_j(T,x) = \sum_{k=1}^{\infty} i\epsilon_j \langle \mu\varphi_j, \varphi_k \rangle \int_0^T v(t) e^{i(\lambda_k - \lambda_j)t} dt \varphi_k(x) e^{-i\lambda_k T}, \forall j \in \{1,...,N\}.$$

Therefore, if $N > 1$, then this linearized system is not controllable : it misses exactly $(N-1)(N+1)$ (real) directions (in the two senses) because

$$\frac{\langle \Phi_{k_1}, \psi_{k_2}(T) \rangle}{\epsilon_{k_1}} = -\frac{\overline{\langle \Phi_{k_2}, \psi_{k_1}(T) \rangle}}{\epsilon_{k_2}}, \forall k_1 \neq k_2 \in \{1,...,N\},$$

$$\Im \frac{\langle \Phi_j(T), \psi_j(T) \rangle}{\epsilon_j \langle \mu\varphi_j, \varphi_j \rangle} = \Im \frac{\langle \Phi_1(T), \psi_1(T) \rangle}{\epsilon_1 \langle \mu\varphi_1, \varphi_1 \rangle}, \forall j \in \{2,...,N\}.$$

These $(N-1)$ last directions are less important than the previous $N(N-1)$ ones because they can be tackled with a 'fictitious' control as in [134]. Thus, let us focus on the $N(N-1)$ first ones. The classical strategy consists in trying to recover them on the second order terms and to use the same strategy as in [53]. However, the present situation is worse than the one considered in the reference [53] because some missed directions 'turn' at the same speed (when no control is applied) : for instance, with $N \geqslant 3$, the missed directions $\langle \phi_2(t), \varphi_1 \rangle$ (which is linked to $\langle \phi_1(t), \varphi_2 \rangle$) and $\langle \phi_3(t), \varphi_1 \rangle$ (which is linked to $\langle \phi_1(t), \varphi_3 \rangle$) 'turn' as $e^{-i\lambda_1 t}$. Therefore the subtle arguments of [53], relying on rotations in the complex plane are not sufficient to conclude : new ideas need to be introduced.

Chapitre 4

Stabilization of Schrödinger équations with discrete spectrum (A6, A15) [32, 33]

This chapter is organized as follows. In Section 4.1, we present the context, a review of previous results and the usual technics. In Section 4.2, we present the article **(A6)** [32] and in Section 4.3, we present the article **(A15)** [33].

4.1 Introduction

The previous chapter dealt with the controllability problem : given two states (an initial one and a final one), we looked for an open loop control ; this control depends on time and on the two given states, but not on the state during the evolution of the control system. In many practical situations, one prefers closed loop controls, i.e. controls which do not depend on the initial state, but depend, at time t, on the state $\psi(t)$, and that asymptotically stabilizes the point one wants to reach. Usually, such closed loop controls (or feedback laws) are more robust to small disturbances. The main issue of this chapter is the feedback stabilization of the ground state, for bilinear Schrödinger PDEs with bilinear control,

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\Delta\psi(t,x) + V(x)\psi(t,x) - u(t)\mu(x)\psi(t,x), x \in \Omega, \\ \psi(t,x) = 0, x \in \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \in \mathbb{N}^*$, $\mu, V : \Omega \to \mathbb{R}$ are smooth functions, and u is a real valued time depending function. The goal is to design explicit feedback laws $u = u(\psi)$ stabilizing asymptotically the ground state.

This problem has been first addressed for finite dimensional models in [134]

$$i\frac{dZ}{dt} = H_0 Z + u(t) H_1 Z$$

where $Z \in \mathbb{C}^N$, H_0 and H_1 are hermitian $N \times N$ matrices. In [134], the control design relies on control Lyapunov functions, the feedback laws are explicit and the convergence proof relies on the LaSalle invariance principle. This reference deals with the situation where the linearized system around the ground state is controllable. A degenerate case is studied in **(A3)** [28], by introducing implicit feedback laws.

The goal of this section is to adapt the results of [134, 28] to PDE models. Indeed, the LaSalle invariance principle is a powerful tool to prove the asymptotic stability of an equilibrium for a finite dimensional dynamic system. However, using it for infinite dimensional systems is more difficult (because closed and bounded subsets are not necessarily compact).

In this chapter, we propose two possible adaptations. A first possible adaptation consists in proving *approximate* convergence results, as for example in **(A6)** [32, 131] ; this strategy is presented in Section 4.2. A second possible adaptation consists in proving a *weak* convergence, as, for example, in [18] and in **(A15)** [33] ; this strategy is presented in Section 4.3.

Actually there exists a third possible adaptation, which consists in proving a *strong* convergence, as for example in [67]. In this case, one needs an additional compactness property for the trajectories of the closed loop system. Another strategy consists in designing strict Lyapunov functions, as for example in [68]. These last two strategies have not been yet applied successfully to bilinear Schrödinger equations.

Finally, let us remark that the feedback stabilization of a quantum system necessitates more complicated models, taking into account the measurement backaction on the system (see for example [132]). The kind of strategy considered in this chapter can be helpful in practice for the open-loop control of quantum systems. Indeed, one can apply the stabilization techniques for the Schrödinger equation in numerical simulations and retrieve the control signal that will be then applied in open-loop on the real physical system.

4.2 Approximate stabilization of a 1D bilinear Schrödinger equation [32] (A6)

4.2.1 Equation and result

The article [32] deals with the equation

$$\begin{cases} i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} - u(t)x\Psi(t,x), x \in (-1/2, 1/2), \\ \Psi(t, \pm 1/2) = 0, \end{cases} \tag{4.2}$$

where $\mu : (-1/2, 1/2) \to \mathbb{R}$ is a smooth function.

For $\sigma \in \mathbb{R}$, we introduce the operator A_σ defined by

$$D(A_\sigma) := (H^2 \cap H^1_0)(I; \mathbb{C}), \quad A_\sigma\varphi := -\frac{1}{2}\frac{\partial^2\varphi}{dx^2} - \sigma x\varphi.$$

Let $(\phi_{k,\sigma})_{k\in\mathbb{N}^*}$ be the eigenvectors of A_σ

$$\phi_{k,\sigma} \in H^2 \cap H^1_0(I, \mathbb{C}), \quad A_\sigma\phi_{k,\sigma} = \lambda_{k,\sigma}\phi_{k,\sigma}$$

where $(\lambda_{k,\sigma})_{k\in\mathbb{N}^*}$ is a non decreasing sequence of real numbers. For $s > 0$ and $\sigma \in \mathbb{R}$, small enough so that $\lambda_{1,\sigma} > 0$, we define

$$H^s_{(\sigma)}(I, \mathbb{C}) := D(A_\sigma^{s/2}),$$

equipped with the norm

$$\|\varphi\|_{H^s_{(\sigma)}} := \left(\sum_{k=1}^{\infty} \lambda^s_{k,\sigma} |\langle\varphi, \phi_{k,\sigma}\rangle|^2 \right)^{1/2},$$

where the symbol $\langle .,. \rangle$ denotes the usual hermitian product of $L^2(I, \mathbb{C})$ i.e.

$$\langle \varphi, \xi \rangle := \int_I \varphi(x)\overline{\xi(x)}dx.$$

For $k \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}$, we define

$$\mathcal{C}_{k,\sigma} := \{\phi_{k,\sigma} e^{i\theta}; \theta \in [0, 2\pi)\}.$$

In order to simplify the notations, we write ϕ_k, λ_k, $\mathcal{C}_k$ instead of $\phi_{k,0}$, $\lambda_{k,0}$, $\mathcal{C}_{k,0}$. We have

$$\lambda_k = \frac{k^2\pi^2}{2}, \quad \phi_k = \begin{cases} \sqrt{2}\cos(k\pi x), & \text{when } k \text{ is odd}, \\ \sqrt{2}\sin(k\pi x), & \text{when } k \text{ is even}. \end{cases} \tag{4.3}$$

When $k \in \mathbb{N}^*$, $\sigma \in \mathbb{R}$, $\Psi_0 = \phi_{k,\sigma}$ and $u \equiv \sigma$, the solution of (4.2) is given by

$$\Psi_{k,\sigma}(t,x) := \phi_{k,\sigma}(x) e^{-i\lambda_{k,\sigma} t}.$$

This function is called the k^{th} eigenstate associated to $u \equiv \sigma$. In the particular case $k = 1$, $\Psi_{1,\sigma}$ is the ground state associated to $u \equiv \sigma$.

The goal of the article **(A6)** [32] is the study of the stabilization of the system (4.2) around the eigenstates $\Psi_{k,\sigma}$. More precisely, for $k \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}$ small, we propose explicit feedback laws $u = u_{k,\sigma}(\Psi)$ such that, for any solution of (4.2) with $u(t) = u_{k,\sigma}(\Psi(t))$, the quantity

$$\limsup_{t \to +\infty} \operatorname{dist}_{L^2(I,\mathbb{C})}(\Psi(t), \mathcal{C}_{k,\sigma})$$

is arbitrarily small. For simplicity sakes, we will only work with the ground state $\Psi_{1,\sigma}$. However, the whole arguments remain valid for the general case.

The main result of the article [32] is the following one.

Theorem 18 *Let $\Gamma > 0$, $s > 0$, $\epsilon > 0$, $\gamma \in (0,1)$. There exists $\sigma^{**} = \sigma^{**}(\Gamma, s) > 0$ such that, for every $\sigma \in (-\sigma^{**}, \sigma^{**})$, there exists a feedback law $v_{\sigma,\Gamma,s,\epsilon,\gamma}(\Psi)$ such that, for every $\Psi_0 \in \mathbb{S} \cap (H^2 \cap H^1_0 \cap H^s_{(\sigma)})(I, \mathbb{C})$ with*

$$\|\Psi_0\|_{H^s_{(\sigma)}} \leqslant \Gamma \text{ and } |\langle \Psi_0, \phi_{1,\sigma} \rangle| > \gamma,$$

the solution of (4.2) with $\Psi(0) = \Psi_0$ and $u(t) = \sigma + v_{\sigma,\Gamma,s,\epsilon,\gamma}(\Psi(t))$ has a unique strong solution, moreover, this solution satisfies

$$\limsup_{t \to +\infty} dist_{L^2}(\Psi(t), \mathcal{C}_{1,\sigma}) \leqslant \epsilon.$$

For $\sigma \neq 0$, the feedback law is given explicitly. For $\sigma = 0$, the feedback law is given by an implicit formula. Theorem 18 provides **semi-global stabilization**. Indeed, any initial condition $\Psi_0 \in \mathbb{S}$ such that $\Psi_0 \in H^s(I, \mathbb{C})$ for some $s > 0$ and $\langle \Psi_0, \phi_{1,\sigma} \rangle \neq 0$ can be moved approximately to the circle $\mathcal{C}_{1,\sigma}$, thanks to an appropriate feedback law. The assumption "$\Psi_0 \in H^s(I, \mathbb{C})$, for some $s > 0$ "is not necessary for doing that, but, when Ψ_0 only belongs to $\mathbb{S}$, then, we need the feedback law to be also a function of the initial state Ψ_0.

4.2.2 Heuristic

As in Theorem 18, let us consider $\Gamma > 0$, $s > 0$, $\epsilon > 0$,$\gamma > 0$, $\sigma \in \mathbb{R}$. First, we consider the case $\sigma \neq 0$. Let $\Psi_0 \in H^s_{(0)}(I, \mathbb{C})$ with

$$\|\Psi_0\|_{H^s_{(0)}} \leqslant \Gamma \text{ and } |\langle \Psi_0, \phi_{1,\sigma}\rangle| \geqslant \gamma.$$

There exists $N = N(\Gamma, s, \epsilon, \gamma) \in \mathbb{N}^*$ such that

$$\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma}\rangle|^2 < \tfrac{\epsilon\gamma^2}{1-\epsilon}. \tag{4.4}$$

Let us consider the Lyapunov function

$$\mathcal{V}(\Psi) = 1 - |\langle \Psi \mid \phi_{1,\sigma}\rangle|^2 - (1-\epsilon)\sum_{k=2}^{N} |\langle \Psi \mid \phi_{k,\sigma}\rangle|^2. \tag{4.5}$$

Note that, this Lyapunov function depends on the constants Γ, s, ϵ, γ through the choice of the cut-off dimension, N. This Lyapunov function encodes two tasks : 1- it prevents the L^2-mass lost through the high-energy eigenstates ; 2- it privileges the increase of the population in the first eigenstate.

When Ψ solves (4.2) with some control $u = \sigma + v$, we have

$$\frac{d\mathcal{V}}{dt} = -2v(t)\Im\Big(\sum_{k=1}^{N} a_k \langle x\Psi \mid \phi_{k,\sigma}\rangle \langle \phi_{k,\sigma} \mid \Psi\rangle\Big),$$

where

$$a_1 := 1 \text{ and } a_k := 1-\epsilon \text{ for } k = 2, ..., N \tag{4.6}$$

and, for $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z.

Thus, the feedback law

$$v(\Psi) := \Im\Big(\sum_{k=1}^{N} a_k \langle x\Psi \mid \phi_{k,\sigma}\rangle \langle \phi_{k,\sigma} \mid \Psi\rangle\Big), \tag{4.7}$$

trivially ensures the decrease of the Lyapunov function (4.5). In the article [32], we prove that the solution of (4.2) with initial condition Ψ_0 and control $u = \sigma + v(\Psi)$ satisfies

$$\limsup_{t\to+\infty} \operatorname{dist}_{L^2}(\Psi(t), \mathcal{C}_{1,\sigma})^2 \leqslant \epsilon. \tag{4.8}$$

This is proved by studying the $L^2(I, \mathbb{C})$-weak limits of $\Psi(t)$ when $t \to +\infty$. Namely, let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $t_n \to +\infty$. Since $\|\Psi(t_n)\|_{L^2(I,\mathbb{C})} \equiv 1$, there exists $\Psi_\infty \in L^2(I, \mathbb{C})$ such that, up to a subsequence, $\Psi(t_n) \to \Psi_\infty$ weakly in $L^2(I, \mathbb{C})$. Using the controllability of the linearized system around $\Psi_{1,\sigma}$ (which is equivalent to $\langle \phi_{1,\sigma}, x\phi_{k,\sigma}\rangle \neq 0$ for every $k \in \mathbb{N}^*$), we are able to prove that $\Psi_\infty = \beta\phi_{1,\sigma}$, where $\beta \in \mathbb{C}$ and $|\beta|^2 \geqslant 1-\epsilon$. This implies (4.8). Note that, the controllability of the linearized system around the trajectory $\Psi_{1,\sigma}$ plays a crucial role here. This is why the developed techniques for $\sigma \neq 0$ can not be applied, directly, to the case $\sigma = 0$.

Now, let us study the case $\sigma = 0$. As emphasized above, the previous strategy does not work for the approximate stabilization of Ψ_1 because the linearized system around Ψ_1 is not controllable. The idea is thus to use the above feedback design (4.7) with a dynamic $\sigma = \sigma(t)$ that converges to zero as $t \to +\infty$ as in [28]. Formally, the convergence toward $\mathcal{C}_{1,\sigma(t)}$ must happen at a faster rate than that of σ toward zero.

In this aim, we consider the Lyapunov function

$$\mathcal{V}(\Psi) = 1 - (1-\epsilon)\sum_{k=1}^{N} |\left\langle \Psi \mid \phi_{k,\sigma(\Psi)} \right\rangle|^2 - \epsilon |\left\langle \Psi \mid \phi_{1,\sigma(\Psi)} \right\rangle|^2, \tag{4.9}$$

where the function $\Psi \mapsto \sigma(\Psi)$ is implicitly defined as below

$$\sigma(\Psi) = \theta\left(\mathcal{V}(\Psi)\right), \tag{4.10}$$

for a slowly varying real function θ. We claim that such a function $\sigma(\Psi)$ exists. When Ψ solves (4.2) then, we have

$$\begin{aligned} \tfrac{d\mathcal{V}}{dt} = \; & -2v(\Psi)\Im\Big(\sum_{k=1}^{N} a_k \left\langle x\Psi \mid \phi_{k,\sigma(\Psi)} \right\rangle \left\langle \phi_{k,\sigma(\Psi)} \mid \Psi \right\rangle \Big) \\ & -\tfrac{d\sigma(\Psi)}{dt} 2\Re\Big(\textstyle\sum_{k=1}^{N} a_k \langle \Psi, \phi_{k,\sigma(\Psi)} \rangle \langle \tfrac{d\phi_{k,\sigma(\Psi)}}{d\sigma}, \Psi \rangle \Big). \end{aligned}$$

where $(a_k)_{1 \leqslant k \leqslant N}$ is defined by (4.6) and the notation

$$\frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}$$

means the derivative of the map $\sigma \mapsto \phi_{k,\sigma}$ taken at the point $\sigma = \sigma(\Psi)$. By definition of $\sigma(\Psi)$, we have

$$\frac{d\sigma(\Psi)}{dt} = \theta'(\mathcal{V}(\Psi))\frac{d\mathcal{V}}{dt}.$$

Thus, the feedback law $u(\Psi) := \sigma(\Psi) + v(\Psi)$ where

$$v(\Psi) := \Im\Big(\sum_{k=1}^{N} a_k \left\langle x\Psi \mid \phi_{k,\sigma(\Psi)} \right\rangle \left\langle \phi_{k,\sigma(\Psi)} \mid \Psi \right\rangle \Big)$$

ensures

$$\frac{d\mathcal{V}}{dt} = -2\mu v(\Psi)^2,$$

where

$$\frac{1}{\mu} = 1 - 2\theta'(\mathcal{V}(\Psi))\Re\Big(\sum_{k=1}^{N} a_k \langle \Psi, \phi_{k,\sigma(\Psi)} \rangle \langle \frac{d\phi_{k,\sigma(\Psi)}}{d\sigma}, \Psi \rangle \Big)$$

is a positive constant, when $\|\theta'\|_{L^\infty}$ is small enough. Thus $t \mapsto \mathcal{V}(\Psi(t))$ is not increasing.

In the article [32], we prove that the solution of (4.2) with initial condition Ψ_0 and control $u = \sigma(\Psi) + v(\Psi)$ satisfies

$$\limsup_{t \to +\infty} \mathrm{dist}_{L^2}(\Psi(t), \mathcal{C}_1)^2 \leqslant \epsilon. \tag{4.11}$$

Again, this is proved by studying the $L^2(I, \mathbb{C})$-weak limits of $\Psi(t)$ when $t \to +\infty$.

4.3 Weak stabilization of bilinear Schrödinger equations [33] (A15)

4.3.1 Equation, heuristic

In the article [33], we study the system (4.1). In a previous work [137], Nersesyan proposed explicit feedback laws and proved the existence of a sequence of times $(t_n)_{n\in\mathbb{N}}$ for which the values of the solution of the closed loop system converge weakly in H^2 to the ground state. In the article [33], we prove the convergence of the whole solution, as $t \to +\infty$.

Let us recall the stabilization strategy proposed in [137]. We introduce the Lyapunov function

$$\mathcal{V}(z) := \alpha \|(-\Delta+V)P_{1,V}z\|^2 + 1 - |\langle z, e_{1,V}\rangle|^2, \quad z \in S \cap H^1_0 \cap H^2,$$

where $\alpha > 0$, $(e_{k,V})_{k\in\mathbb{N}^*}$ are the eigenvectors of the operator $-\Delta+V$, $(-\Delta+V)e_{k,V} = \lambda_{k,V}e_{k,V}$ and $P_{1,V}z := z - \langle z, e_{1,V}\rangle e_{1,V}$ is the orthogonal projection in L^2 onto the closure of $\mathrm{Span}\{e_{k,V}, k \geq 2\}$. Notice that $\mathcal{V}(z) \geq 0$ for all $z \in S \cap H^1_0 \cap H^2$ and $\mathcal{V}(z) = 0$ if and only if $z = ce_{1,V}, |c| = 1$. For any $z \in S \cap H^1_0 \cap H^2$, we have

$$\mathcal{V}(z) \geq \alpha\|(-\Delta+V)P_{1,V}z\|^2 \geq \frac{\alpha}{2}\|\Delta(P_{1,V}z)\|^2 - C_1 \geq \frac{\alpha}{4}\|\Delta z\|^2 - C_2.$$

Thus

$$C(1+\mathcal{V}(z)) \geq \|z\|_2 \tag{4.12}$$

for some constant $C > 0$. We wish to choose a feedback law $u(\cdot)$ such that

$$\frac{d}{dt}\mathcal{V}(z(t)) \leq 0$$

for the solution $z(t)$ of (4.1). Let us assume that $\Delta z(t) \in H^1_0 \cap H^2$ for all $t \geq 0$. Using (4.1), we get

$$\begin{aligned}
\tfrac{d}{dt}\mathcal{V}(z(t)) &= 2\alpha\Re\Big[\langle(-\Delta+V)P_{1,V}\dot{z}, (-\Delta+V)P_{1,V}z\rangle\Big] - 2\Re\Big[\langle \dot{z}, e_{1,V}\rangle\langle e_{1,V}, z\rangle\Big] \\
&= 2\alpha\Re\Big[\langle(-\Delta+V)P_{1,V}(i\Delta z - iVz - iu\mu z), (-\Delta+V)P_{1,V}z\rangle\Big] \\
&\quad -2\Re\Big[\langle i\Delta z - iVz - iu\mu z, e_{1,V}\rangle\langle e_{1,V}, z\rangle\Big].
\end{aligned}$$

Integrating by parts and using the fact that V is real valued and

$$(-\Delta+V)P_{1,V}z|_{\partial D} = z|_{\partial D} = e_{1,V}|_{\partial D} = 0,$$

we obtain

$$\begin{aligned}
&2\alpha\Re\Big[\langle -i(-\Delta+V)^2P_{1,V}z, (-\Delta+V)P_{1,V}z\rangle\Big] - 2\Re\Big[\langle i\Delta z - iVz, e_{1,V}\rangle\langle e_{1,V}, z\rangle\Big] \\
= \ &2\alpha\Re\Big[\langle -i\nabla(-\Delta+V)P_{1,V}z, \nabla(-\Delta+V)P_{1,V}z\rangle\Big] \\
&+2\alpha\Re\Big[\langle -iV(-\Delta+V)P_{1,V}z, (-\Delta+V)P_{1,V}z\rangle\Big] \\
&+2\lambda_{1,V}\Re\Big[\langle iz, e_{1,V}\rangle\langle e_{1,V}, z\rangle\Big] = 0.
\end{aligned}$$

Thus

$$\frac{d}{dt}\mathcal{V}(z(t)) = 2u\Im\Big[\alpha\langle(-\Delta+V)P_{1,V}(\mu z),(-\Delta+V)P_{1,V}z\rangle - \langle\mu z, e_{1,V}\rangle\langle e_{1,V}, z\rangle\Big].$$

Let us take $u(z)$ defined by

$$u(z) := -\delta\Im\Big[\langle\alpha(-\Delta+V)P_{1,V}(\mu z),(-\Delta+V)P_{1,V}z\rangle - \langle\mu z, e_{1,V}\rangle\langle e_{1,V}, z\rangle\Big], \tag{4.13}$$

where $\delta > 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(z(t)) = -\frac{2}{\delta}u^2(z(t)), \tag{4.14}$$

thus $t \mapsto \mathcal{V}(z(t))$ is not increasing and one may expect that $z(t) \to \mathcal{C} := \{ce_{1,V}, c \in \mathbb{C}, |c| = 1\}$, in some sense, when $t \to +\infty$. We consider the closed loop system

$$i\dot{z} = -\Delta z + V(x)z + u(z)\mu(x)z, \quad x \in D. \tag{4.15}$$

The local (in time) well posedness in $H_0^1 \cap H^2$ of the Cauchy problem associated is standard (see [52]). Moreover, from the construction of the feedback law $u(z)$ it follows that a finite-time blow-up in $H_0^1 \cap H^2$ is impossible. Hence the solution is global in time.

4.3.2 Main result

Let us introduce the following condition on the functions V and Q.

Condition 1 *The functions $V, \mu \in C^\infty(\overline{D}, \mathbb{R})$ are such that :*

(i) $\langle\mu e_{1,V}, e_{j,V}\rangle \neq 0$ for all $j \geq 2$,

(ii) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $j, p, q \geq 1$ such that $\{1, j\} \neq \{p, q\}$ and $j \neq 1$.

See the papers [138, 129] for the proof of genericity of this condition. The below theorem is the main result of this article

Theorem 19 *Let $\mathcal{U}_t$ be the resolving operator of the closed loop system (4.15), (4.13). Under Condition 1, there is a finite or countable set $J \subset \mathbb{R}_+^*$ such that for any $\alpha \notin J$ and $z_0 \in S \cap H_0^1 \cap H^2$ with $0 < \mathcal{V}(z_0) < 1$ we have*

$$\mathcal{U}_t(z_0) \rightharpoonup \mathcal{C} \text{ in } H^2 \text{ as } t \to \infty, \tag{4.16}$$

where $\mathcal{C} := \{ce_{1,V} : c \in \mathbb{C}, |c| = 1\}$.

Remark 7 *This Theorem proves the semi-global stabilization of the ground state. Indeed, for every $z_0 \in S \cap H_0^1 \cap H^2$ such that $z_0 \notin \mathcal{C}$, one may chose $\alpha = \alpha(\|z_0\|_2) > 0$ small enough so that the condition $0 < \mathcal{V}(z_0) < 1$ is fulfilled.*

The first step of the proof consists in checking that the LaSalle invariance set locally coincides with $\mathcal{C}$, which is proved in [137]. Then, we prove the continuity of the solution of the closed loop system, with respect to initial conditions, for the weak H^2-topology. The key point of this proof is that the feedback law $u(z)$ is well defined for z strictly less regular than H^2 (formally, $z \in H^{3/2}$ is sufficient). Finally, we prove that for every $\alpha \notin J$ and $z_0 \in S \cap H_0^1 \cap H^2$ with $0 < \mathcal{V}(z_0) < 1$, the weak H^2 ω-limit set of $\{\mathcal{U}_t(z_0); t \geqslant 0\}$ is contained in $\mathcal{C}$.

Generalizations with different regularities are possible : with Lyapunov functions inspired by the H^s distance to the target, one may prove weak H^s stabilization. Generalizations to other bilinear equations (for instance wave equations) are possible.

4.4 Conclusion, open problems, perspectives

4.4.1 Weak stabilization on unbounded domains

In [33] **(A15)**, our proof uses compact injections between Sobolev spaces on a bounded domain. Thus, the stabilization of bilinear Schrödinger equations when such compact injections cannot be used is still an open problem.

4.4.2 Strong stabilization of bilinear Schrödinger equations

In Section 4.3, we have presented a weak H^2-stabilization result. An open problem is : do the same feedback laws ensure strong convergence ? For the proof of such a result, one would need to prove the compactness in H^2 of the trajectories of the closed loops system. Such a result, if it holds, seems difficult, because of the important coupling between the different frequencies. This strategy has been used in [67], but, in this reference, an almost diagonal form is favorable for the proof of the precompactness of the trajectories. Such a configuration is far from being true in our case : there is an important coupling between the frequencies.

4.4.3 Simultaneous stabilization of N Schrödinger equations with one control

Let us consider the system

$$\begin{cases} i\frac{\partial \phi_j}{\partial t}(t,x) = (-\Delta + V)\phi_j(t,x) - \epsilon_j u(t)\mu(x)\phi_j(t,x), x \in \Omega, j \in \{1,...,N\}, \\ \phi_j(t,x) = 0, x \in \partial\Omega, j \in \{1,...,N\}, \end{cases} \tag{4.17}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n, N \in \mathbb{N}^*$, $V, \mu : \Omega \to \mathbb{R}$ are smooth functions, and ϵ_1,...,ϵ_N are different real numbers. It is a control system in which

- the state is $(\phi_1, ..., \phi_N)$ with $\|\phi_j(t)\|_{L^2(\Omega)} = 1$ for $j = 1, ..., N$ and for every t,
- the control is the real valued function u.

This control problem is related to the control of molecular dynamics [162] and the quantum gate generation in information theory [152].

Let us try to stabilize the N first eigenstates, i.e. to realize (up to phase factors)

$$(\phi_1, ..., \phi_N)(t) \to (\varphi_1, ..., \varphi_N) \text{ when } t \to +\infty.$$

We consider the new Lyapunov function which is the sum of the N H^2-distances between $\phi_k(t)$ and $\psi_k(t)$ for $k = 1, ..., N$. Working as in [33] **(A15)**, we get a feedback law $u = u(\phi_1, ..., \phi_N)$ ensuring the decrease of this Lyapunov function along the trajectories of the closed loop system.

In this situation, the strategy of proof of [33] **(A15)**, does not work because the invariant set does not coincide with the target. This is due to the fact that, in the feedback law, the factors of $e^{i(\lambda_k - \lambda_K)t}$ (for $k \neq K \in \{1, ..., N\}$) come from ψ_k and ψ_K simultaneously and they can compensate each other. For the same reason, the linearized system around the trajectory $(\psi_1, ..., \psi_N, u = 0)$ is not controllable : it misses at least $(N + 1)(N - 1)$ real directions (see Section 3.7.3).

This situation has to be compared with the finite dimensional situation. Indeed, in finite dimension, the following results are known.

Every linear control system which is controllable can be asymptotically stabilized by means of continuous stationary feedback laws. This result implies that, if the linearized system of a given nonlinear control system is controllable, then this nonlinear control system can be locally asymptotically stabilized by means of continuous stationary feedback laws.

When the nonlinear control system is controllable, but its linearized system is not, it is not always possible to stabilize it by means of continuous stationary feedback laws (a counterexample is the system

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \\ \dot{x}_3 = x_1 u_2 - x_2 u_1, \end{cases}$$

studied in [65, Example 11.2 page 289]). To overcome this impossibility, two main strategies have been proposed in the literature :

– discontinuous feedback laws,
– continuous time varying feedback laws (or periodic feedback laws) as, for instance in [60].

In our case, the controllability of the nonlinear system may be proved with power series expansions (this work is under progress, see Section 3.7.3). Therefore, the previous two strategies (discontinuous or time-varying feedback laws) may be tried to realize the stabilization of system (4.17). Notice that perturbations and implicit feedback laws, as in [28, 32], are useless : here the 'multiple eigenvalues' stay multiple even if we perturb the Hamiltonian. Another possible strategy consists in changing, in a subtle way, the Lyapunov function, in order to get the desired invariant set.

4.4.4 Stabilization of nonlinear Schrödinger equation

The validity of this Lyapunov approach has not been investigated yet for nonlinear Schrödinger equations with bilinear control, for instance,

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) \pm |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), \\ \frac{\partial\psi}{\partial x}(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0 \end{cases}$$

Even a 'weak' stabilization result (for a sequence of times $(t_n)_{n\in\mathbb{N}}$, as in [138]). would be interesting. Indeed, coupled with the local exact controllability result of [31], it may provide the global exact controllability of this nonlinear system.

Chapitre 5

Genericity and spectral control for a linear Schrödinger equation (A9) [26]

5.1 Introduction

Equation : In the article **(A9)** [26], we study the systems

$$\begin{cases} i\frac{\partial \Psi}{\partial t}(t,x) = -\Delta\Psi(t,x) - \langle v(t), \mu(x)\rangle \psi_1(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.1}$$

$$\begin{cases} i\frac{\partial \Psi}{\partial t}(t,x) = -\Delta\Psi(t,x) - \langle v(t), \mu(x)\rangle \psi_1(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \Psi(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ \dot{d}(t) = s(t), \\ \dot{s}(t) = v(t), \end{cases} \tag{5.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \in \{2,3\}$, $\mu : \Omega \to \mathbb{R}^N$ is a regular function, $\psi_1(t,x) := \varphi_1(x) e^{-i\lambda_1 t}$ is the ground state. The system (5.1) (resp. (5.2)) is a linear control system in which

- the state is the function Ψ (resp. the couple (Ψ, s, d)) where $\Psi(t) \in T_{\mathbb{S}}(\psi_1(t))$ for every $t \in \mathbb{R}_+$,
- the control is $u : t \in \mathbb{R} \to \mathbb{R}^N$.

Here, we have used the notation

$$T_{\mathbb{S}}\varphi := \left\{ \xi \in L^2(\Omega, \mathbb{C}); \Re\left(\int_\Omega \xi(q)\overline{\varphi(q)}dq \right) = 0 \right\}.$$

The system (5.1) is the linearized system of a bilinear Schrödinger equation around the ground state. The system (5.2) corresponds to the model of a quantum particle in a moving box, for which the command is the acceleration of the box (see [27]). We study the spectral controllability of this system.

Motivation : In 2D the exact controllability of (5.1) is a difficult problem Indeed, it is equivalent to the solvability of a trigonometric moment problem in which the frequencies are the eigenvalues of the Dirichlet-Laplacian on Ω. A necessary and sufficient condition for the existence of a solution in $L^2(0,T)$ of this trigonometric moment problem is the existence of a uniform gap (in a weak sense, see [164]) between the frequencies. However, the existence of such a gap, for the eigenvalues of the Dirichlet-Laplacian, on a generic 2D bounded domain, is an open problem.

Since the exact controllability is a difficult problem, it is natural to investigate weaker controllability properties. Here, we consider the spectral controllability, which corresponds to the exact controllability between any finite sums of eigenvectors of the Dirichlet-Laplacian operator.

5.2 Results

First, let us recall the definition of spectral controllability, for the systems (5.1) and (5.2).

Definition 4 (Spectral controllability for (5.1)) *The system (5.1) is spectral controllable in time T if, for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the solution of (5.1) with $\Psi(0) = \Psi_0$ satisfies $\Psi(T) = \Psi_f$, where*

$$\mathcal{D} := Span\{\phi_k; k \in \mathbb{N}^*\}.$$

For the system (5.2), this definition needs to be adapted because of the presence of s and d in the state variable and because the directions $\Im\langle\Psi(t), \psi_1(t)\rangle$ and $s(t)$ are linked. Indeed, any solution of (5.2) with $(\Psi, s, d)(0) = (\Psi_0, s_0, d_0)$ satisfies

$$\Im\langle\Psi(t), \psi_1(t)\rangle = \Im\langle\Psi_0, \psi_1(0)\rangle + \sum_{j=1}^{n}\langle\mu^{(j)}\phi_1, \phi_1\rangle[s^{(j)}(t) - s^{(j)}(0)], \forall t, \tag{5.3}$$

where, for $x \in \mathbb{R}^n$, $x^{(j)}$ denotes its components, $x = (x^{(1)}, ..., x^{(n)})$ and $\langle.,.\rangle$ denotes the $L^2(\Omega, \mathbb{C})$-scalar product. Therefore, we study the following controllability property for (5.2).

Definition 5 (Spectral controllability for (5.2)) *The system (5.2) is spectral controllable in time T if for every $\Psi_0 \in \mathcal{D} \cap T_S\psi_1(0)$, $\Psi_f \in \mathcal{D} \cap T_S\psi_1(T)$ with $\Im\langle\Psi_f, \psi_1(T)\rangle = \Im\langle\Psi_0, \psi_1(0)\rangle$, for every $d_0 \in \mathbb{R}^n$, there exists $v \in L^2((0,T),\mathbb{R}^n)$ such that the solution of (5.2) with $(\Psi, s, d)(0) = (\Psi_0, 0, d_0)$ satisfies $(\Psi, s, d)(T) = (\Psi_f, 0, 0)$.*

In order to state our results, we first give several definitions relative to the domain and the dipolar moment.

Definition 6 (Kalman condition, (Kal)) *Let Ω be a domain of $\mathbb{R}^n$, $n = 2, 3$ with C^1 boundary. Then Ω verifies Property (Kal) if*

(Kal) any eigenvalue λ of $-\Delta_\Omega^D$ has a multiplicity $m \leq n$ and the vectors $\langle\mu\phi_1, \phi_{k_1}\rangle$,...,$\langle\mu\phi_1, \phi_{k_m}\rangle$ are linearly independent in $\mathbb{R}^n$, where $k_1 < ... < k_m$ and $\phi_{k_1}, ..., \phi_{k_m}$ are the eigenvectors associated to λ.

Definition 7 (Simplicity of the spectrum, $(Simp)$) *Let Ω be a domain of $\mathbb{R}^n$, $n = 2, 3$ with C^1 boundary. Then Ω verifies Property $(Simp)$ if*

$$(Simp) \quad \text{the eigenvalues of } -\Delta_\Omega^D \text{ are simple.}$$

Definition 8 (Non zero projection, $(NonZ)$**)** *Consider* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$*,* $n = 2, 3$ *and* $(\phi_k)_{k\in\mathbb{N}^*}$ *the complete orthonormal system of eigenvectors of* $-\Delta_\Omega^D$*. Then* $\mu\phi_1$ *has a non zero projection on* $(\phi_k)_{k\in\mathbb{N}^*}$ *if, for every integer* $k \geq 2$*, we have*

$$(NonZ)_k \quad \langle \mu\phi_1, \phi_k \rangle \neq 0.$$

In that case, we say that μ *verifies Property* $(NonZ)$*.*

Remark that if a domain Ω satisfies $(Simp)$, then condition (Kal) reduces to condition $(NonZ)$. The next theorem gathers our result regarding the spectral controllability properties for system (5.1).

Theorem 20 ***(1)*** *Let* Ω *be a domain of* $\mathbb{R}^2$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ *verifying* (Kal)*. Then, there exists* $T_{min} = T_{min}(\Omega) > 0$ *such that*

(1.a) *for every* $T > T_{min}$*, system (5.1) is spectral controllable in time* T *;*

(1.b) *for every* $T < T_{min}$*, system (5.1) is not spectral controllable in time* T*, under the additional assumption*

$$\mu(x) = \tilde{\mu}(x)e_1 \text{ where } \tilde{\mu} \in C^0(\overline{\Omega}, \mathbb{R}). \tag{5.4}$$

(2) *Let* Ω *be a domain of* $\mathbb{R}^n$*,* $n = 2, 3$*, with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ *such that* (Kal) *is not verified. Then, system (5.1) is not spectral controllable.*

(3) *Let* Ω *be a domain of* $\mathbb{R}^3$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$ *of the form (5.4). Then, system (5.1) is not spectral controllable.*

Remark 8 *Let us emphasize that* (Kal) *holds true generically with respect to the pair* (Ω, μ) *because conditions* $(Simp)$ *and* $(NonZ)$ *hold true simultaneously generically with respect to the pair* (Ω, μ)*, where* Ω *is a domain of* $\mathbb{R}^2$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$*. Indeed, the genericity of* $(Simp)$ *with respect to the domain* Ω *is a classical result. Moreover, for a domain* Ω *of* $\mathbb{R}^2$ *with* C^1 *boundary verifying* $(Simp)$*, the set*

$$\{\mu \in C^0(\overline{\Omega}, \mathbb{R}^2); \langle \mu\phi_1, \phi_k \rangle \neq 0, \forall k \in \mathbb{N}^*\}$$

is dense in $C^0(\overline{\Omega}, \mathbb{R}^2)$ *(it can be proved thanks to Baire's Lemma).*

As for system (5.2), we prove the following result.

Theorem 21 ***(1)*** *Let* Ω *be a domain of* $\mathbb{R}^2$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^2)$ *verifying* (Kal)*. Let* $T_{min} = T_{min}(\Omega)$ *be as in Theorem 20. Then, system (5.2) is spectral controllable in time* $T > T_{min}$*.*

(2) *Let* Ω *be a domain of* $\mathbb{R}^n$*,* $n = 2, 3$*, with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^n)$ *such that* (Kal) *is not verified. Then, system (5.2) is not spectral controllable.*

(3) *Let* Ω *be a domain of* $\mathbb{R}^3$ *with* C^1 *boundary and* $\mu \in C^0(\overline{\Omega}, \mathbb{R}^3)$*. Then system (5.2) is not spectral controllable : for every* $T > 0$ *and* $m \in \mathbb{N}^*$*, there exists* $d_0 \in \mathbb{R}^3$ *such that* $(i\phi_m, 0, d_0)$ *is not zero controllable in time* T*.*

Remark 9 *Notice that in Item* ***(3)*** *of Theorem 21, the dipolar moment* μ *is not necessarily one dimensional. Thus, we prove a stronger non controllability result for this 3D system, than the one given in Theorem 20* ***(3)****. This improvement is due to the presence of* s *and* d *in the state variable.*

The proof of Theorems 20 and 21 relies on a previous work by Haraux and Jaffard. In [100] they prove that a family of complex exponentials

– is minimal in $L^2(0,T)$ if $T > 2\pi d$,

– is not minimal in $L^2(0,T)$ if $T < 2\pi d$,

where d is the density of the frequencies in the Weyl formula. The proof of the statement ***(3)*** of Theorem 21 involves different ideas, from complex analysis, about the set of zeros of holomorphic functions, as in [57].

From Theorem 21, one knows that, in 2D, property (Kal) is a necessary and sufficient condition for the spectral controllability of (5.1) and (5.2) in large time. We next use that characterization to prove that spectral controllability of (5.1) and (5.2) in large time holds true generically with respect to the 2D domain Ω. For that purpose, let us first precise the topology on domains we are using, then define genericity and finally state the condition on the dipolar moment μ that ensures the genericity.

For $l \geq 1$, the set $\mathbb{D}_l$ of domains Ω of $\mathbb{R}^2$ with C^l boundary. Following [153], we define next a topology on $\mathbb{D}_l$. Consider the Banach space $W^{l+1,\infty}(\Omega,\mathbb{R}^2)$ equipped with its standard norm. For $\Omega \in \mathbb{D}_l$, $u \in W^{l+1,\infty}(\Omega,\mathbb{R}^2)$, let $\Omega + u := (\mathrm{Id}+u)(\Omega)$ be the subset of points $y \in \mathbb{R}^2$ such that $y = x + u(x)$ for some $x \in \Omega$ and $\partial\Omega + u := (\mathrm{Id}+u)(\partial\Omega)$ its boundary. For $\varepsilon > 0$, let $V(\Omega,\epsilon)$ be the set of all $\Omega + u$ with $u \in W^{l+1,\infty}(\Omega,\mathbb{R}^2)$ and $\|u\|_{W^{l+1,\infty}} \leq \varepsilon$. The topology of $\mathbb{D}_l$ is defined by taking the sets $V(\Omega,\varepsilon)$ with ε small enough as a base of neighborhoods of Ω. Then, $\mathbb{D}_l$ is a Banach space.

Definition 9 *We say that a property (P) is generic in $\mathbb{D}_l$ if the set of domains of $\mathbb{D}_l$ on which this property holds true is dense in $\mathbb{D}_l$: for every $\Omega \in \mathbb{D}_l$, there exists $\rho > 0$ such that the set $\{u \in E_\rho(\Omega); \Omega + u$ satisfies $(P)\}$ is dense in $E_\rho(\Omega)$, where $E_\rho(\Omega) := \{u \in W^{l+1,\infty}(\Omega,\mathbb{R}^2); \|u\|_{W^{l+1,\infty}} < \rho\}$.*

Definition 10 (Non locally constant, (NLC)) *A map $\mu \in C^0(\mathbb{R}^2,\mathbb{R}^2)$ is said to be nowhere locally constant if, for every $\mu_0 \in \mathbb{R}^2$, the level set $\{q \in \mathbb{R}^2 \quad \mu(q) = \mu_0\}$ has an empty interior.*

Note that if μ is (NLC) and continuously differentiable, then the subset of $\mathbb{R}^n$, $n = 2, 3$, where the differential of μ is not zero, must be open and dense.

We now state one of the main results of the paper.

Theorem 22 *Let $\mu \in C^1(\mathbb{R}^2,\mathbb{R}^2)$. The spectral controllability in large time for system (5.2) is generic in $\mathbb{D}_3$ if and only if μ is nowhere locally constant.*

According to Item ***(2)*** of Theorem 21, the proof of the previous theorem reduces to establishing the next proposition, since $(Simp)$ and $(NonZ)$ both verified imply that (Kal) holds true.

Proposition 2 *Let $\mu \in C^1(\mathbb{R}^2,\mathbb{R}^2)$. If $\Omega \in \mathbb{D}_1$, we say that Ω has property (A) if $(Simp)$ and $(NonZ)$ hold true for Ω. Then, property (A) is generic in $\mathbb{D}_3$ if and only if μ is nowhere locally constant.*

For the proof of Proposition 2, the Baire Lemma allows a first reduction of the problem. However, standard technics relying on shape differentiation are not sufficient to conclude.

Thus, we also use a careful study of Dirichlet-to-Neumann operators associated to certain Helmholtz equation.

5.3 Conclusion, open problems, perspectives

The article **(A9)** [26] underlines the existence of a positive minimal time, (related to the density of the eigenvalues of the Dirichlet-Laplacian) necessary for the spectral controllability of linear 2D Schrödinger equations. This result may be seen as a hint for the behavior of 2D bilinear Schrödinger equations. We conjecture that, on generic 2D bounded domains Ω, there exists a time $T_{min}(\Omega) > 0$ such that,

- for every $T < T_{min}(\Omega)$, the bilinear Schrödinger equation (3.35) is not exactly controllable in a strong sense (the reachable set is a non flat submanifold of some functional space)
- for every $T > T_{min}(\Omega)$, the bilinear Schrödinger equation (3.35) is exactly controllable in some functional space.

We refer to the section 3.7.2 for a more detailed discussion.

Chapitre 6

Control and stabilization of the Bloch equation, a bilinear system with continuous spectrum (A10, A14) [29, 30]

6.1 Introduction

The Bloch equation represents an ensemble of non interacting spins, in a magnetic field

$$B(t) = (u(t), v(t), 1),$$

with a dispersion in their Larmor frequency $\omega \in (\omega_*, \omega^*)$. Each spin is represented by a unitary vector $M = M(t, \omega) \in \mathbb{S}^2$ whose time evolution is ruled by the equation

$$\frac{\partial M}{\partial t}(t, \omega) = [u(t)e_1 + v(t)e_2 + \omega e_3] \wedge M(t, \omega), \omega \in (\omega_*, \omega^*), \tag{6.1}$$

where $-\infty < \omega_* < \omega^* < +\infty$, (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, $\wedge$ is the vectorial product on $\mathbb{R}^3$. It is an infinite dimensional bilinear control system, in which

- the state is the function M,
- the control is the couple $(u, v) : [0, T] \to \mathbb{R}^2$.

Thus, we want to control simultaneously a continuum of ordinary differential equations (with parameter $\omega \in (\omega_*, \omega^*)$).

This problem has been introduced and studied in [124, 123, 125], by Li and Khaneja, who introduced the notion of 'ensemble controllability' (which is the approximate controllability in $L^2((\omega_*, \omega^*), \mathbb{S}^2)$) In these references, the authors highlight, for three common dispersions in NMR spectroscopy, the role of Lie algebras and non-commutativity in the design of a compensating control sequence and consequently in the characterization of ensemble controllability. These pioneer articles provide convincing arguments indicating why the system (6.1) should be ensemble controllable with unbounded controls. In this chapter, we provide several rigorous mathematical results in this direction.

Another interest of the Bloch equation is that it may be seen as a toy model for infinite dimensional bilinear systems with continuous spectrum. Indeed, the spectrum of the operator $\mathcal{A}$ defined by

$$(\mathcal{A}M)(\omega) := \omega e_3 \wedge M(\omega),$$

is formally $-i(\omega_*,\omega^*) \cup i(\omega_*,\omega^*)$: for every $\omega_\sharp \in (\omega_*,\omega^*)$, the eigenvector associated to $\pm i\omega_\sharp$ is $(1,\mp i,0)^t \delta_{\omega_\sharp}(\omega)$. Thus, the study of the Bloch equation is also motivated by the understanding of the controllability of bilinear Schrödinger equations with continuous spectrum.

Indeed, most controllability results available for infinite dimensional bilinear systems are related to systems with discrete spectra (see for instance, Chapter 2 for exact controllability results, [54, 137] and Section 4 for approximate controllability results). As far as we know, very few controllability studies consider systems admitting a continuous part in their spectra. In [131] an approximate controllability result is given for a system with mixed discrete/continuous spectrum : the Schrödinger partial differential equation of a quantum particle in an N-dimensional decaying potential is shown to be approximately controllable (in infinite time) to the ground bounded state when the initial state is a linear superposition of bounded states.

This chapter is organized as follows. In Section 6.2, we study the controllability of the Bloch equation with open loop controls. In Section 6.3, we investigate the feedback stabilization of this equation.

6.2 Controllability (A10) [29]

The article **(A10)** [29] is organized as follows.

In a first step, we study the exact controllability of the nonlinear system (6.1), locally around the reference trajectory $(M \equiv e_3, u \equiv 0)$, in finite time. First, we prove that the simultaneous exact controllability with respect to ω in the whole space $\mathbb{R}$ (i.e. $\omega_* = -\infty$, $\omega^* = +\infty$) does not hold with a priori bounded controls. Indeed, for every time $T > 0$, the reachable set, from $M_0 \equiv e_3$, with a priori bounded controls in $L^2(0,T)$, is a non flat submanifold of the functional space $L^2 \cap C^0_b(\mathbb{R})$, with infinite codimension. Then, with an analyticity argument, we deduce that the simultaneous exact controllability with respect to ω in a bounded interval (ω_*,ω^*), $-\infty < \omega_* < \omega^* < +\infty$, does not hold neither.

Since the exact controllability of (6.1) is impossible with a priori bounded controls, we investigate the exact controllability of (6.1) with unbounded controls.

In a second step, completing the arguments of [124, 123, 125], we prove the ensemble controllability of (6.1) with unbounded controls : any initial condition $M_0 : (\omega_*,\omega^*) \to \mathbb{S}^2$ in $H^1((\omega_*,\omega^*),\mathbb{S}^2)$ can be steered approximately in $L^2(\omega_*,\omega^*)$ to e_3. This approximate controllability indeed holds for stronger norms, for instance $\|.\|_{L^\infty}$ and $\|.\|_{H^s}$, $\forall s \in (0,1)$. The controls used to realize this motion are sequences of pulses presented in [124] (but one may also use controls in $L^\infty_{loc}([0,+\infty))$, by smoothing the previous ones) and the proof relies on non-commutativity and functional analysis.

Finally, in a last step, we propose explicit unbounded controls realizing the asymptotic local (exact) controllability to e_3, simultaneously with respect to ω in a bounded interval. Here, the proof relies on Fourier analysis.

6.2.1 Non exact controllability with a priori bounded controls

Let us study the reachable set from $M(0) \equiv e_3$ for (6.1) with bounded controls $(u, v) \in L^2((0,T), \mathbb{R}^2)$. Let x, y, z be the components of the solution M of (6.1) :

$$M(t,\omega) = (x(t,\omega), y(t,\omega), z(t,\omega))^T,$$

$Z(t,\omega) := (x+iy)(t,\omega)$ and $w(t) := (v - iu)(t)$. Notice that, when $M(0) \equiv e_3$ and w is small enough in $L^1(0,T)$, then $z(t,\omega) > 0$ for every $(t,\omega) \in (0,T) \times \mathbb{R}$ and

$$Z(t,\omega) = -\int_0^t w(\tau)\sqrt{1 - |Z(\tau,\omega)|^2} e^{-i\omega\tau} d\tau e^{i\omega t}, \forall (t,\omega) \in [0,T] \times \mathbb{R}. \tag{6.2}$$

In a first step, we take $\omega_* = -\infty$, $\omega^* = +\infty$. Let us precise the functional framework in which (6.2) is well posed.

Proposition 3 *Let $T > 0$ and $R := 1/(2\sqrt{T})$. For every $w \in L^2(0,T)$ with $\|w\|_{L^2(0,T)} < R$, there exists a unique $Z \in C^0([0,T], L^2(\mathbb{R})) \cap C^0_b([0,T] \times \mathbb{R})$ solution of (6.2) and it satisfies*

$$\|Z\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant \sqrt{T}\|w\|_{L^2(0,T)}, \quad \|Z\|_{C^0([0,T],L^2(\mathbb{R}))} \leqslant 2\sqrt{2\pi}\|w\|_{L^2(0,T)}.$$

We prove that the reachable set from zero, with bounded controls (u, v), for (6.2) in $L^2((0,T), \mathbb{R}^2)$ is a non flat submanifold of the functional space $L^2 \cap C^0_b(\mathbb{R})$, with infinite codimension. In particular, (6.1) is not locally controllable with a priori bounded controls (u, v) in $L^2((0,T), \mathbb{R}^2)$.

Theorem 23 *Let $T > 0$ and $R := 1/(4\sqrt{3T})$. The image of the end point map*

$$\begin{array}{rccl} F_T: & B_R[L^2(0,T)] & \to & L^2 \cap C^0_b(\mathbb{R}) \\ & w & \mapsto & Z(T,.) \text{ where } Z \text{ solves } (6.2), \end{array} \tag{6.3}$$

is a strict submanifold of $L^2 \cap C^0_b(\mathbb{R})$ of infinite codimension that does not coincide with its tangent space at zero.

The proof of this Theorem relies on the implicit function theorem : $dF_T(0)$ is injective because it is the Fourier transform.

In a second step, let us go back to the physical case $-\infty < \omega_* < \omega^* < +\infty$. We prove the following result.

Theorem 24 ***(i)*** *Let $T > 0$, $u, v \in L^2(0,T)$ and M be the solution of*

$$\begin{cases} \frac{\partial M}{\partial t}(t,\omega) = [u(t)e_1 + v(t)e_2 + \omega e_3] \wedge M(t,\omega), & (t,\omega) \in (0,T) \times \mathbb{C}, \\ M(0,\omega) = e_3. \end{cases} \tag{6.4}$$

Then $\omega \in \mathbb{C} \mapsto Z(T,\omega)$ is holomorphic.

(ii) *Let $T > 0$ and $R := 1/(4\sqrt{3T})$. There exists $Z_f : (\omega_*, \omega^*) \to \mathbb{C}$ analytic such that, for every $\epsilon^* > 0$, there exists $\epsilon \in (0, \epsilon^*)$ such that, for every $w \in B_R[L^2(0,T)]$, the solution of (6.1) satisfies $Z(T) \neq \epsilon Z_f$.*

As a consequence, there are arbitrarily small analytic targets on (ω_*, ω^*) that cannot be reached exactly in finite time, with controls having a prescribed L^2-bound.

6.2.2 Approximate controllability with unbounded controls

Since the Bloch equation (6.1) is not controllable with a priori bounded controls, it is natural to investigate its controllability with unbounded controls and we will use controls that are finite sums of Dirac masses.

First, let us introduce the definition of solutions of (6.1) with Dirac controls. This definition requires the generators of rotations around the axis x, y and z

$$\Omega_x := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega_y := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Omega_z := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Definition 11 *Let $b \in [0,+\infty)$, $\beta, \gamma \in \mathbb{R}$ and $M_0 : (\omega_*, \omega^*) \to \mathbb{S}^2$. The solution of (6.1) with $M(0) = M_0$, $u(t) = \beta\delta_b(t)$, $v(t) = \gamma\delta_b(t)$ is*

$$M(t,\omega) = \begin{cases} \exp(\omega\Omega_z t)M_0(\omega) \text{ for } t \in [0,b), \\ \exp(\omega\Omega_z(t-b))\exp(\beta\Omega_x + \gamma\Omega_y)\exp(\omega\Omega_z b)M_0(\omega) \text{ for } t \in (b,+\infty), \end{cases}$$

i.e.

$$M(b^+,\omega) = \exp(\beta\Omega_x + \gamma\Omega_y)M(b^-,\omega).$$

Thus, a Dirac mass located at the time b acts on the state as an instantaneous rotation on the whole profile of M. This definition is motivated by the following convergence

$$\lim_{\epsilon\to 0} U\Big[b+\epsilon; \frac{\beta}{\epsilon}1_{[b,b+\epsilon]}, \frac{\gamma}{\epsilon}1_{[b,b+\epsilon]}, .\Big] = U[b^+; \beta\delta_b, \gamma\delta_b, .].$$

where $U[t;u,v,M_0]$ is the value at time t, of the solution of (6.1), with initial condition M_0 at time 0. Let us introduce the set D of finite sums of Dirac masses on $[0,+\infty)$. We prove the following result.

Theorem 25 *Let $M_0 \in H^1((\omega_*,\omega^*),\mathbb{S}^2)$. There exist $(t_n)_{n\in\mathbb{N}} \in [0,+\infty)^{\mathbb{N}}$, and $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ such that*

$$U[t_n^+; u_n, v_n, M_0] \to e_3 \text{ weakly in } H^1((\omega_*,\omega^*),\mathbb{R}^3).$$

By smoothing the controls u_n and v_n, one easily get the same statement with $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \in L^\infty_{loc}([0,+\infty),\mathbb{R})^{\mathbb{N}}$. Thanks to the compactness of the injection $H^1(\omega_*,\omega^*) \to L^2(\omega_*,\omega^*)$ (resp. $H^1(\omega_*,\omega^*) \to L^\infty(\omega_*,\omega^*)$), Theorem 25 gives the approximate controllability of (6.1), to e_3, for the $L^2((\omega_*,\omega^*),\mathbb{R}^3)$-norm (resp. the $L^\infty(\omega_*,\omega^*)$-norm), in finite time.

The proof of Theorem 25 is in two steps. The first step is the following result already presented in [124, 125].

Proposition 4 *Let $P, Q \in \mathbb{R}[X]$. The flow of (6.1) can generate*

$$I + \tau[P(\omega)\Omega_x + Q(\omega)\Omega_y] + o(\tau) \text{ when } \tau \to 0,$$

with controls that are finite sums of Dirac masses. More precisely, for every $\epsilon > 0$, there exists $\tau^* = \tau^*(P,Q,\epsilon) > 0$ such that, for every $\tau \in [0,\tau^*]$, there exist $T > 0$ and $u, v \in D$, such that

$$\Big\| U[T^+; u, v, .] - \Big(I + \tau[P(\omega)\Omega_x + Q(\omega)\Omega_y]\Big)\Big\|_{\mathcal{L}(H^1((\omega_*,\omega^*),\mathbb{R}^3), H^1((\omega_*,\omega^*),\mathbb{R}^3))} \leqslant \epsilon\tau.$$

The second step of the proof consists in deducing from Proposition 4 the following Lemma.

Lemma 2 *Let $M \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ be such that $M' \neq 0$. There exist $T > 0$, $u, v \in D$ such that*

– *one has*

$$\|U[T^+; u, v, M]'\|_{L^2} < \|M'\|_{L^2},$$

– *for every sequence $(M_n)_{n\in\mathbb{N}} \in H^1((\omega_*, \omega^*), \mathbb{S}^2)^{\mathbb{N}}$ satisfying*

$$\|M_n'\|_{L^2} \leqslant \|M'\|_{L^2}, \forall n \in \mathbb{N} \tag{6.5}$$

and

$$M_n \to M \text{ weakly in } H^1((\omega_*, \omega^*), \mathbb{R}^3) \tag{6.6}$$

there exists an extraction φ such that

$$\|U[T^+; u, v, M_{\varphi(n)}]'\|_{L^2} \leqslant \|M_{\varphi(n)}'\|_{L^2}, \forall n \in \mathbb{N}.$$

Then, we prove Theorem 25 in the following way. Let us fix $M_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ be such that $M_0 \neq e_3$ (otherwise $t_n \equiv 0$ gives the conclusion). We consider the set K constituted by weak H^1 limits of points on trajectories starting from M_0

$$\begin{aligned} K := \; & \{\widetilde{M} \in H^1((\omega_*, \omega^*), \mathbb{S}^2);\, \exists (t_n)_{n\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}, \exists (u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}} \\ & \text{ such that } \|U[t_n^+; u_n, v_n, M_0]'\|_{L^2} \leqslant \|M_0'\|_{L^2}, \forall n \in \mathbb{N} \\ & \text{ and } U[t_n^+; u_n, v_n, M_0] \to \widetilde{M} \text{ weakly in } H^1((\omega_*, \omega^*), \mathbb{R}^3)\} \end{aligned}$$

and the quantity

$$m := \inf\{\|\widetilde{M}'\|_{L^2};\, \widetilde{M} \in K\}.$$

Notice that K is not empty because it contains M_0 (take $t_n \equiv 0$). Classical arguments allow to prove the existence of $e \in K$ such that $m = \|e'\|_{L^2}$. Thanks to Lemma 2, we prove that $m = 0$ (otherwise, one may decrease more). Therefore, K contains at least one constant. By applying an additional rotation, this constant may be e_3.

6.2.3 Asymptotic exact controllability with unbounded controls

The result of the previous section is interesting because it ensures the global approximate controllability to e_3 in L^2. However, this result is purely theoretical and does not provide any control realizing this task. Thus, we also propose explicit controls realizing the asymptotic exact controllability to $-e_3$, locally around $-e_3$.

First, let us introduce some notations. For a function $f : (-\pi, \pi) \to \mathbb{C}$, we denote by $c_n(f)$ its Fourier coefficients and by $N(f)$ their l^1-norm :

$$c_n(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) e^{-in\omega} d\omega, \quad N(f) := \sum_{n\in\mathbb{Z}} |c_n(f)|.$$

For a function $f : (0, \pi) \to \mathbb{C}$, we define

$$c_n(f) := c_n(\tilde{f}), \forall n \in \mathbb{Z}, \quad N(f) := N(\tilde{f}),$$

where $\tilde{f} : (-\pi, \pi) \to \mathbb{C}$, $\tilde{f}(\omega) := f(|\omega|)$. For a vector valued map $M = (x, y, z) : (0, \pi) \to \mathbb{R}^3$, we define $N(M) := N(x) + N(y) + N(z)$ and $Z =: x + iy$.

We prove the following result.

Theorem 26 *There exists $\delta > 0$ such that, for every $M_0 : [0, \pi] \to \mathbb{S}^2$ with $N[Z_0] < \delta$ and $z_0 < -1/2$, there exists $\epsilon = \epsilon(M_0) > 0$ such that, the solution of (6.1) with $M(0) = M_0$,*

$$u(t) := \frac{\pi}{\epsilon} 1_{[k,k+\epsilon]}(t) - \sum_{p=1}^{2k-1} \Im\Big(c_{-k+p}(Z_0)\Big)\frac{1}{\epsilon} 1_{[k+p,k+p+\epsilon]}(t) + \frac{\pi}{\epsilon} 1_{[3k,3k+\epsilon]}(t),$$

$$v(t) := -\sum_{p=1}^{2k-1} \Re\Big(c_{-k+p}(Z_0)\Big)\frac{1}{\epsilon} 1_{[k+p,k+p+\epsilon]}(t),$$

where $k = k(M_0) \in \mathbb{N}$ is such that

$$\sum_{|n|>k} |c_n(Z_0)| < \frac{N(Z_0)}{4}, \tag{6.7}$$

satisfies

$$N[Z(3k+\epsilon)] < \frac{N[Z_0]}{2},$$

$$z(3k+\epsilon) < -1/2.$$

By iterating this process, we find an increasing sequence $(t_n)_{n\in\mathbb{N}} \in [0, +\infty)^{\mathbb{N}}$ and two controls $u, v \in L^\infty_{loc}([0, +\infty), \mathbb{R})$ such that

$$N[Z(t_n)] < \frac{1}{2^n} N[Z_0].$$

Thus, $\|M(t_n) + e_3\|_{L^\infty} \to 0$ when $n \to +\infty$. These explicit controls provide the exact asymptotic controllability to e_3.

The main idea of the proof consists in noticing that, for the linearized system around e_3 (and equivalently around $-e_3$), one can cancel the Fourier coefficients of the initial condition associated to negative frequencies, by choosing appropriate sums of Dirac masses (located at positive integer times) for controls. The same manipulation is also possible, approximately, for the nonlinear system.

We start close to $-e_3$. The 'main' Fourier components of the initial condition are associated to the frequencies $n \in [-k, k]$. Our control strategy consists in

- waiting for a time k, which realizes a shift to the right of all the Fourier components of the solution (now, the 'main' Fourier coefficients are associated to the frequencies $n \in [0, 2k]$),
- moving to $-e_3$ by applying a Dirac mass of amplitude π on u, this reverses the Fourier coefficients (now the 'main' Fourier coefficients are associated to the frequencies $n \in [-2k, 0]$),
- once we are near $-e_3$, we cancel all the Fourier coefficients of the solution, associated to frequencies $n \in [-2k, 0]$ (now, the only remaining Fourier coefficients are approximately those of the initial condition associated to the frequencies $|n| > k$, thus the 'size' of the solution has been divided by 2).

6.3 Stabilization (A14) [30]

In the article **(A14)** [30], we propose an explicit feedback law that stabilizes asymptotically the Bloch equation around a uniform state of spin +1/2 or -1/2. The convergence holds locally in $H^1(\omega_*, \omega^*)$ and for the weak H^1 topology. The proof relies on an adaptation of the LaSalle invariance principle to infinite dimensional systems. Numerical simulations illustrate the efficiency of these feedback laws, even for initial conditions far from the equilibrium.

6.3.1 The impulse-train control to reduce dispersion

In view of the controls used in the previous section, it is natural to consider a control with the following "impulse-train" structure

$$u = \tilde{u} + \sum_{k=1}^{+\infty} \pi\ \delta(t - kT), \quad v = (-1)^{\mathcal{E}\left(\frac{t}{T}\right)} \tilde{v} \tag{6.8}$$

for some period $T > 0$, which is fixed in all the article ($\mathcal{E}(\gamma)$ denotes the integer part of the real number γ). The new controls $\tilde{u}$ and $\tilde{v}$ belong to $L^1_{loc}(\mathbb{R})$. Then, after each impulse that is applied at time $t = kT$, x remains unchanged, but y and z are moved to their opposites, that is

$$(x, y, z)(kT^+) = (x, -y, -z)(kT^-).$$

Let $\varsigma = (-1)^{\mathcal{E}\left(\frac{t}{T}\right)}$. Considering the following change of variable

$$(x, y, z)(t, \omega) \mapsto \Big(x(t, \omega), \zeta(t) y(t, \omega), \zeta(t) z(t, \omega)\Big)$$

one gets the following dynamics

$$\begin{cases} \dot{x} = -\varsigma \omega y + \tilde{v} z, \\ \dot{y} = \varsigma \omega x - \tilde{u} z, \\ \dot{z} = -v x + u y, \end{cases} \tag{6.9}$$

with the new control $(\tilde{u}, \tilde{v})$ as in (6.8). It is as if, between $[kT, (k+1)T]$ and $[(k+1)T, (k+2)T]$, one is changing the sign of ω, but the solution remains continuous in t (but not differentiable in t at the instants $t = kT, k \in \mathbb{N}$). In other words, the application of the impulses at $t = kT$ changes the sense of rotation of the null input solution. One would expect that this impulse-train control is reducing the average dispersion of the solution. Roughly speaking, the dispersion observed for the open-loop system (6.1) with (u, v) as input is strongly reduced and almost canceled for the open-loop system (6.9) with $(\tilde{u}, \tilde{v})$ as input.

6.3.2 Heuristics of the Lyapunov-like control

Now let $Z(t, \omega)$ and $\Omega(t)$ defined by

$$Z := x + iy, \quad \Omega := \tilde{v} - i\tilde{u}.$$

Then one may write (6.9) in the form

$$\begin{cases} \dot{Z}(t, \omega) &= i\varsigma(t)\omega Z(t, \omega) + \Omega(t) z(t, \omega), \\ \dot{z}(t, \omega) &= -\Re\left[\Omega(t)\overline{Z(t, \omega)}\right], \end{cases}$$

where $\Re(\xi)$ (resp. $\overline{\xi}$) denotes the real part (resp. the complex conjugate number) of a complex number $\xi \in \mathbb{C}$. It is easy to see that the following transformation

$$\tilde{Z}(t,\omega) = Z(t,\omega)e^{-i\omega\int_0^t \varsigma(\tau)d\tau}$$

converts the system into the driftless form

$$\left\{ \begin{array}{rcl} \dot{Z}(t,\omega) & = & \Omega(t)z(t,\omega)e^{-i\omega\int_0^t \varsigma} \\ \dot{z}(t,\omega) & = & -\Re\left[\Omega(t)\overline{Z(t,\omega)}e^{-i\omega\int_0^t \varsigma}\right] \end{array} \right. \tag{6.10}$$

where, for notation simplicity, one lets $Z(t,\omega)$ stand for $\tilde{Z}(t,\omega)$, and one lets $\int_0^t \varsigma$ stand for $\int_0^t \varsigma(\tau)d\tau$.

Now consider the following Lyapunov-like functional :

$$\mathcal{L} = \frac{1}{2}\int_{\omega_*}^{\omega^*} \left\{|Z'|^2 + (z')^2 + 2Gz\right\} d\omega \tag{6.11}$$

where G is a positive real number, $Z' := \frac{\partial Z}{\partial \omega}$, $z' := \frac{\partial z}{\partial \omega}$. The function $\mathcal{L}$ is minimal when $M = -e_3$, i.e. $Z \equiv 0$ and $z \equiv -1$. One may write

$$\frac{d}{dt}\mathcal{L}(t) = \Re\left(\int_{\omega_*}^{\omega^*} \left\{\bar{Z}'\dot{Z}' + z'\dot{z}' + G\dot{z}\right\} d\omega\right) = \Re\left[\Omega(t)H(t)\right] \tag{6.12}$$

where

$$H(t) := \int_{\omega_*}^{\omega^*} \left\{i\left(\int_0^t \varsigma\right)\left(\bar{Z}z' - \bar{Z}'z\right) - G\bar{Z}\right\} e^{-i\omega\int_0^t \varsigma} d\omega.$$

Hence one may take $\Omega(t) = -K_p\bar{H}(t)$, where K_p is a positive real number, obtaining

$$\Omega(t) = K_p\int_{\omega_*}^{\omega^*} \left\{i\left(\int_0^t \varsigma\right)\left(Zz' - Z'z\right) + GZ\right\} e^{i\omega\int_0^t \varsigma} d\omega. \tag{6.13}$$

It follows that

$$\frac{d\mathcal{L}}{dt}(t) = -\frac{1}{K_p}|\Omega(t)|^2.$$

The closed loop system is well posed in $H^1((\omega_*,\omega^*),\mathbb{S}^2)$: the local (in time) well posedness is classical, and no explosion is possible in finite time, thanks to the choice of the feedback law.

6.3.3 Convergence result

Theorem 27 *There exists $\delta' > 0$ such that, for every $M_0 \in H^1((\omega_*,\omega^*),\mathbb{S}^2)$ with $\|M_0 + e_3\|_{H^1} \leq \delta'$,*

$$M(t) \to -e_3 \text{ weakly in } H^1 \text{ when } t \to +\infty.$$

The proof of this result is in two steps. In a first step, we prove that, locally, the invariant set coincides with the target $\{e_3\}$. In a second step, we prove the convergence by adapting the LaSalle invariance principle (periodic version) to this infinite dimensional setting. As in [33] **(A15)**, the key point of the proof is that the feedback law (6.13) is well defined for M strictly less regular than H^1 ($H^{1/2}$ is sufficient). Finally, for the weak stabilization, the presence of a continuous spectrum does not influence the proof.

6.3.4 No global stabilization

Now, let us explain why these feedback laws may not provide global stabilization in $H^1((\omega_*, \omega^*), \mathbb{S}^2)$.

The first obstruction is a topological one : the space $H^1((\omega_*, \omega^*), \mathbb{S}^2)$ cannot be continuously deformed to one point (because $\mathbb{S}^2$ is not), thus global stabilization in this space is impossible.

Actually, for our explicit feedback laws, it is easy to see that $M^0 \equiv +e_3$ is an invariant solution. It is interesting to know whether it is the only one (i.e. if one may expect the stabilization of any initial condition $M^0 \neq e_3$). Actually, it is not the case : the invariant set contains an infinite number of non trivial functions. Moreover, all its elements may be written explicitly.

6.4 Conclusion, open problems, perspectives

In **(A10)** [29], we prove the non exact controllability with a priori bounded L^2-controls, in finite time, because the reachable set from e_3 is a non flat submanifold of the functional space $L^2 \cap C^0_b(\mathbb{R})$, with infinite codimension. The equation of this submanifold and the validity of the same negative result in infinite time (i.e. the non asymptotic exact controllability to e_3 with bounded controls) are open problems.

In **(A10)** [29], we prove the exact controllability to e_3 with unbounded controls, in infinite time. The validity of the same result in finite time is also open.

Concerning the feedback stabilization studied in **(A14)** [30], several problems are still open. Are the invariant solutions stable or unstable ? Does the local stabilization hold for the strong H^1-topology (not only the weak one) ? Is it possible to get semi-global stabilization ? What is the value of convergence rates ? Is it possible to realize arbitrarily fast stabilization ? Is this strategy possible for tracking problems ?

Chapitre 7

Hyperbolic systems (A11) [35]

7.1 Introduction

Equations : In this article, we consider partially dissipative hyperbolic systems. In a first part, we study the asymptotic behavior of constant coefficient linear systems

$$\frac{\partial w}{\partial t}(t,x) + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j}(t,x) = -Bw(t,x) \tag{7.1}$$

where $A_1, ..., A_m$ are $n \times n$ real symmetric matrices and B is an $n \times n$ matrix of the form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, D \in \mathbb{R}^{n_2 \times n_2}, X^t D X > 0, \forall X \in \mathbb{R}^{n_2} - \{0\}, \tag{7.2}$$

where the first diagonal bloc is $n_1 \times n_1$ with $n_1 + n_2 = n$. In a second part, we study balance laws in m space dimension with n-components

$$\frac{\partial w}{\partial t}(t,x) + \sum_{j=1}^{m} \frac{\partial F_j(w)}{\partial x_j}(t,x) = Q(w)(t,x) \tag{7.3}$$

where $Q, F_1, ..., F_m : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions and

$$Q(w) = \begin{pmatrix} 0 \\ q(w) \end{pmatrix}, q(w) \in \mathbb{R}^{n_2}.$$

Previous results : The solutions of the system (7.1) may be expressed explicitly thanks to the Fourier transform

$$\hat{w}(t,\xi) = \exp[E(\xi)t]\hat{w}_0(\xi), \quad E(\xi) := -B - iA(\xi), \quad A(\xi) := \sum_{j=1}^{m} \xi_j A_j.$$

When $n_2 \neq n$, the matrix $E(\xi)$ is not coercive. However, it is well known that the interaction between the dissipation term $-Bw$ and the dynamic of the system may dissipate all the components. Actually, Shizuta et Kawashima proved in [151] that, under the condition

$$\text{(SK)} : \forall \xi \in \mathbb{R}^m, \text{Ker}(B) \cap \{ \text{ eigenvectors of } A(\xi)\} = \{0\},$$

we have

$$\exists C, c > 0 \text{ such that } \exp[E(\xi)t] \leqslant Ce^{-c\min\{1,|\xi|^2\}t}, \forall \xi \in \mathbb{R}^m, \forall t \in [0, +\infty). \tag{7.4}$$

Thanks to this property, one may prove that any solution of (7.1) associated to an initial condition $w^0 \in L^1 \cap L^2(\mathbb{R}^m)$ may be decomposed in the following way

$$w = w_1 + w_2 \text{ where } \|w_1(t)\|_{L^2} \leqslant Ce^{-\lambda t}\|w^0\|_{L^2} \text{ and } \|w_2\|_{L^\infty} \leqslant Ct^{-\frac{m}{2}}\|w^0\|_{L^1}, \forall t \in [0, +\infty). \tag{7.5}$$

In this decomposition, w_1 corresponds to the high frequencies and w_2 to the low frequencies of the initial condition w_0. The condition (SK) is a sufficient condition for the dissipative term to affect all the components of the system. The goal of the first part of the article **(A11)** [35] is to extend these results.

The existence of global smooth solutions of (7.3), in a neighborhood of a constant equilibrium W_e (i.e. $Q(W_e) = 0$), has been proved by Yong in [168], under the (SK) assumption on the linearized system around this equilibrium and under appropriate entropy assumptions (see also [98] for the 1D case). The second goal of the article **(A11)** [35] is to precise this result, thanks to the technics developed in the linear case : we give an explicit lower bound on the size of the neighborhood where global existence holds. Such a result may be useful, for instance, in relaxation (see, for instance [72]).

7.2 Asymptotic behavior of linear systems

In **(A11)** [35], we propose a simpler proof of Shizuta-Kawashima's result. Thanks to this proof, we can also extend the analysis to situations where the (SK) condition does not hold.

7.2.1 Lyapunov function and explicit decay rates

Our study relies on the link between the (SK) condition and the Kalman rank condition, from control theory. Precisely, we have the following result, that may be proved with elementary tools.

Lemma 3 *Let $n \in \mathbb{N}^*$, A, B be $n * n$ matrices with real coefficients, such that B has the form (7.2). The following statements are equivalent*

- ***(1)*** *$\{$eigenvectors of $A\} \cap Ker(B) = \{0\}$,*
- ***(2)*** *$\forall y \in \mathbb{C}^n - \{0\}$, $t \mapsto B\exp(At)y$ does not vanish on $\mathbb{R}$,*
- ***(3)*** *$\forall y \in \mathbb{C}^n - \{0\}$, there exists $k \in \{0, 1, ..., n-1\}$ such that $BA^k y \neq 0$,*
- ***(4)*** *for every $a_0, ..., a_{n-1} > 0$, the expression*

$$N(y) := \left(\sum_{k=0}^{n-1} a_k |BA^k y|^2\right)^{1/2}$$

  defines a norm on $\mathbb{C}^n$
- ***(5)*** *(A, B) satisfies the Kalman rank condition : the $(n^2) * n$ Kalman matrix*

$$K := \begin{pmatrix} B \\ BA \\ \dots \\ BA^{n-1} \end{pmatrix} \tag{7.6}$$

  has rank n.

Now, let us go back to the ordinary differential equation (ODE) solved pointwise by the Fourier transform of the solution. For $\xi \in \mathbb{R}^*$, we denote by $\rho := |\xi|$ and $\omega := \xi/\rho \in \mathbb{S}^{m-1}$.

Proposition 5 *We fix a family of non-negative real numbers $(m_k)_{0\leqslant k\leqslant n}$ such that*

$$0 = m_0 < m_1 < ... < m_n, \tag{7.7}$$

$$m_k - \frac{m_{k-1} + m_{k+1}}{2} \geqslant \delta > 0, \forall k = 1, ..., n-1, \tag{7.8}$$

for some $\delta > 0$.

Let $A_1, ..., A_m, B$ be $n * n$ real matrices such that B has form (7.2). For $\omega \in S^{m-1}$, $\epsilon > 0$ we define the symmetric non-negative matrix

$$M_\epsilon(\omega) := \sum_{k=0}^{n-1} \epsilon^{m_k} (A(\omega)^t)^k B^t B A(\omega)^k \tag{7.9}$$

and its minimal eigenvalue

$$N_{*,\epsilon}(\omega) := \min\{\langle x, M_\epsilon(\omega)x\rangle; x \in S^{n-1}\} = \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\}. \tag{7.10}$$

Then, there exist $\epsilon_* = \epsilon_*(A_1, ..., A_m, B) \in (0,1)$ and $\tilde{c} = \tilde{c}(A_1, ..., A_m, B) > 0$ such that, for every $\epsilon \in (0, \epsilon_*)$, $x_0 \in \mathbb{C}^n$, $\rho \in (0, +\infty)$ and $\omega \in S^{m-1}$, the solution of

$$\dot{x} = -(B + i\rho A(\omega))x, \quad x(0) = x_0 \in \mathbb{C}^n. \tag{7.11}$$

satisfies

$$|x(t)| \leqslant \sqrt{3}|x_0| e^{-\tilde{c} N_{*,\epsilon}(\omega) \min\{1,\rho^2\} t}, \forall t \in (0, +\infty). \tag{7.12}$$

Remark 10 *This result provides an exponential decay rate for* $\exp[E(\rho\omega)t]$, *which is explicit in terms of ρ and ω, and more precisely on $N_{*,\epsilon}(\omega)$. The main interest of this proposition is that it reduces the problem of the asymptotic behavior of the solutions of (7.1) to the study of the real valued map $\omega \in S^{m-1} \mapsto N_{*,\epsilon}(\omega) \in \mathbb{R}_+$.*

*Note that Proposition 5 holds without assuming the (SK) condition, which, in fact, only enters when trying to obtain a uniform lower bound on $N_{*,\epsilon}(\omega)$ for $\omega \in S^{m-1}$: thanks to Lemma 3 (**(1)** $\Leftrightarrow$ **(4)**), the condition (SK) is equivalent to the existence of $N_* > 0$ such that $N(\omega) \geqslant N_*, \forall \omega \in \mathbb{S}^{m-1}$.*

In particular, it could be that, for some value ω^* of ω, $N_{*,\epsilon}(\omega^*) = 0$. In that case, to get explicit decay rates for (7.1), one has to analyze the behavior of $N_{*,\epsilon}(\omega)$ for values ω close to ω^*. This fact will play an important role when deriving decompositions of solutions in the absence of the (SK) condition and analyzing their decay rates as $t \to \infty$.

The proof of Proposition 5 relies on the construction of an explicit strict Lyapunov function for (7.11),

$$\mathcal{L}_{\rho,\omega,\epsilon}(x) := |x|^2 + \min\left\{\rho, \frac{1}{\rho}\right\} \sum_{k=1}^{n-1} \epsilon^{m_k} \Im\langle BA(\omega)^{k-1}x, BA(\omega)^k x\rangle.$$

This construction is inspired by those introduced by Villani in [165, 166] to derive decay estimates for partially diffusive systems. It takes advantage of the link between (SK) and the Kalman rank condition presented in Lemma 3.

7.2.2 L^2-stability and non dissipated solutions

In view of the previous result, it is natural to introduce the set of degeneracy

$$\mathcal{D}(A_1, ..., A_m, B) := \{\xi \in \mathbb{R}^m; \mathrm{Ker}(B) \cap \{\text{eigenvectors of} A(\xi)\} \neq \{0\}\}.$$

It is also the set on which the function $N_{*,\epsilon}(\xi/|\xi|)$ vanishes. The starting point of the study of the L^2-stability of the system (7.1) is the following result.

Proposition 6 *Let $A_1, ..., A_m, B$ be $n * n$ real matrices such that $A_1, ..., A_m$ are symmetric and B has form (7.2). The set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$ is an algebraic submanifold of $\mathbb{R}^m$. In other words, there exists a finite family of polynomials $(P_j)_{j\in J} \subset \mathbb{R}[X]$ such that*

$$\mathcal{D}(A_1, ..., A_m, B) = \{\xi \in \mathbb{R}^m; P_j(\xi) = 0, \forall j \in J\}. \quad (7.13)$$

Thus, either $\mathcal{D}(A_1, ..., A_m, B) = \mathbb{R}^m$, or $\mathcal{D}(A_1, ..., A_m, B)$ has zero Lebesgue measure. Moreover, $\mathcal{D}(A_1, ..., A_m, B)$ is stable by homotheties.

Again, the proof of this proposition relies on Lemma 3 : on the set of degeneracy, the rank of the Kalman matrix is $< n$, thus, any $n * n$ subdeterminant of this matrix vanishes, each subdeterminant provides a polynomial in ξ. When all these polynomials coincide with 0, then $\mathcal{D}(A_1, ..., A_m, B) = \mathbb{R}^m$, otherwise, $\mathcal{D}(A_1, ..., A_m, B)$ is an algebraic submanifold.

Finally, only two situations are possible :

- either $\mathcal{D}(A_1, ..., A_m, B)$ is a strict subset of $\mathbb{R}^m$, and then, all solutions tend to zero in L^2 strongly as $t \to \infty$,
- or $\mathcal{D}(A_1, ..., A_m, B) = \mathbb{R}^m$, and then, there are non-trivial solutions of constant L^2-norm.

The first point is an application of the dominated convergence theorem (the Lebesgue measure of an algebraic submanifold is zero). For the proof of the second point, we construct explicitly non dissipated solutions.

7.2.3 Decomposition of the solutions

As proved in the previous sections, whenever the measure of $\mathcal{D}(A_1, ..., A_m, B)$ vanishes, all solutions tend to zero in L^2 as $t \to \infty$. In this subsection, we analyze the asymptotic behavior in some more detail. As emphasized in Remark 10, the asymptotic behavior of the solutions of (7.1) reduces to the study of the real valued function $\omega \in S^{m-1} \mapsto N_{*,\epsilon}(\omega)$, defined in Proposition 5 ; more precisely, it is closely linked to the way this function vanished in $\mathcal{D}(A_1, ..., A_m, B)$. This study is the key point of this subsection.

In all this section, we will consider situations in which the set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$ has zero measure and is the union of vector subspaces. This assumption is restrictive because, in general, when the measure of $\mathcal{D}(A_1, ..., A_m, B)$ vanishes, this set is an algebraic submanifold (see Proposition 6), which is not necessarily a union of vector subspaces. However, this assumption holds in many particular examples (see [35, Section 3.2]). In this case, we state a decomposition for the solutions of (7.1) under the additional assumption

$$N_{*,\epsilon}(\omega) \geqslant c \mathrm{dist}(\omega, S^{m-1} \cap \mathcal{D})^\alpha \ , \forall \omega \in S^{m-1} \quad (7.14)$$

for some $\alpha \geqslant 2$, $c > 0$. In this decomposition, there are 4 terms

– the high frequencies, far from the set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$, that have the same behavior as w_1 in (7.5),
– the low frequencies, far to the set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$, that have the same behavior as w_2 in (7.5),
– the high frequencies, close to the set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$, that decay more slowly than the two previous ones,
– the low frequencies, close to the set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$, that decay even more slowly than the three previous ones.

Let us state precisely this result. For $g : \mathbb{R}^m \to \mathbb{R}^n$, we define $L(g) : (0, +\infty) \to (0, +\infty]$,

$$L(g)(\rho) := \sup\{|g(\xi)|; \xi \in \mathbb{R}^m, |\xi| = \rho\}.$$

Proposition 7 *We assume $m \geqslant 2$. Let $A_1, ..., A_m, B$ be $n * n$ real matrices with $A_1, ..., A_m$ symmetric and B of the form (7.2). Let $\epsilon \in (0, \epsilon_*)$ where ϵ_* is as in Proposition 1. We assume*
– ***(A1)***

$$\mathcal{D}(A_1, ..., A_m, B) = \cup_{j=1}^{J} \mathcal{D}_j,$$

where $J \in \mathbb{N}^*$ and $\mathcal{D}_j$ is a vector subspace of $\mathbb{R}^m$ with codimension $r_j \in \mathbb{N}^*$,
– ***(A2)** there exists $c_j > 0$, $\alpha_j \geqslant 2$ for $j = 1, ..., J$, such that, for every $\omega \in S^{m-1}$ with $dist(\omega, \mathcal{D}(A_1, ..., A_m, B)) = dist(\omega, \mathcal{D}_j)$,*

$$N_{*,\epsilon}(\omega) \geqslant c_j \, dist(\omega, \mathcal{D}_j)^{\alpha_j}. \tag{7.15}$$

Then, there exists $C = C(A_1, ..., A_m, B)$ and $\lambda = \lambda(A_1, ..., A_m, B) > 0$ such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n)$ with

$$N_1(w^0) := \int_1^{\infty} \rho^{m-1} L[\hat{w}^0](\rho) d\rho < \infty, \tag{7.16}$$

the solution of (7.1) with initial condition w^0 can be decomposed as

$$w = w_1 + w_2 + w_3 + w_4$$

where (7.5) holds and

$$\|w_3(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leqslant \frac{C}{t^{\frac{1}{\alpha}}} N_1(w^0), \forall t \in (0, +\infty), \tag{7.17}$$

where $\alpha := \max\{\alpha_1, ..., \alpha_J\}$ and

$$\|w_4(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leqslant C \left(\sum_{j=1}^{J} \frac{1}{t^{\frac{r_j}{\alpha_j}}} \right) \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}, \forall t \in (0, +\infty). \tag{7.18}$$

The proof of this result relies on a careful study of the Fourier transform of the solution.

Remark 11 *The assumption (7.16) holds, in particular, when w^0 belongs to the the Schwartz space $\mathcal{S}(\mathbb{R}^m, \mathbb{R}^n)$.*

Remark 12 *We refer to the Examples 1 and 2.a of [35, Subsection 3.2] for which the assumption **(A1)** holds with different values for the codimensions r_j.*

Note that the assumption $\alpha_j \geqslant 2$ is natural. Indeed, let us consider $\omega^* \in S^{m-1} \cap \mathcal{D}(A_1, ..., A_m, B)$ and let us assume that the map $\omega \mapsto N_{*,\epsilon}(\omega)$ is smooth in a neighborhood of ω^* (this can be justified, for example, when $N_{*,\epsilon}(\omega^*) = 0$ is a simple eigenvalue of the matrix $M_\epsilon(\omega^*)$ defined by (7.9)). Since $N_{*,\epsilon}$ is non negative and $N_{*,\epsilon}(\omega^*) = 0$, then, necessarily, $dN_{*,\epsilon}(\omega^*) = 0$. Thus, the Taylor formula justifies that, in a neighborhood of ω^* we have

$$N_{*,\epsilon}(\omega) \leqslant |d^2 N_{*,\epsilon}(\omega^*)||\omega - \omega^*|^2.$$

*Therefore, if **(A2)** holds, then, necessarily $\alpha_j \geqslant 2$.*

Remark 13 *Note that in Proposition 7, we do not assume (SK) to hold. Consequently, in the decomposition of the solution, one has two other terms that decay more slowly with rates $t^{-1/\alpha}$ and $\max\{t^{-r_j/\alpha_j}; 1 \leqslant j \leqslant J\}$. Note also that, necessarily, $r^j/\alpha_j < m/2$ because $r_j < m$ and $\alpha \geqslant 2$ whenever $\mathcal{D}(A_1, ..., A_m, B)$ is not trivial.*

Then, in [35], thanks to this general result, we study in details several particular cases.

For instance, when $n_1 = 1$ (i.e. only one direction is not dissipated by B), the set of degeneracy $\mathcal{D}(A_1, ..., A_m, B)$ is always a vector subspace of $\mathbb{R}^m$ and the function $N_{*,\epsilon}$ always satisfies

$$N_{*,\epsilon}(\omega) \geqslant c\mathrm{dist}(\omega, \mathcal{D})^2.$$

Thus, in this case, we have a complete classification of the possible asymptotic behaviors.

Another example is the case $n = 2$ (2-components systems), $n_1 = 1$ (one direction dissipated by B) :

- either $\mathcal{D}(A_1, ..., A_m, B) = \{0\}$, or, equivalently (SK) is satisfied, then, any solution is the sum of an exponentially decaying component ($e^{-\lambda t}$ for some $\lambda > 0$), and another one decaying like $t^{-m/2}$,
- either $\mathcal{D}(A_1, ..., A_m, B)$ is an hyperplane of $\mathbb{R}^m$, then any solution is the sum of 3 components, the two first ones decay as in the previous case, the third one decays like $1/\sqrt{t}$,
- or $\mathcal{D}(A_1, ..., A_m, B) = \mathbb{R}^m$, then any solution is the sum of a (non dissipated) traveling wave and a exponentially decaying component.

In the general case, we conjecture that the function $N_{*,\epsilon}$ satisfies

$$N_{*,\epsilon}(\omega) \geqslant c\mathrm{dist}(\omega, \mathcal{D})^{2(n-1)},$$

which would lead to a general decomposition of the solutions, when $\mathcal{D}(A_1, ..., A_m, B)$ is a union of vector subspaces. However, this is an open problem.

Finally, let us summarize our results in the following array.

m,n	(SK)	$\mathcal{D}$	L^2 stability	decomposition
$m=1$	yes	$\{0\}$	yes	$e^{-t}+\frac{1}{\sqrt{t}}$
$\forall n$	no	$\mathbb{R}$	no	$e^{-t}+\frac{1}{\sqrt{t}}+$ tr. wave
$n=2$	yes	$\{0\}$	yes	$e^{-t}+\frac{1}{t^{\frac{m}{2}}}$
$\forall m$	no	hyperplane	yes	$e^{-t}+\frac{1}{t^{\frac{m}{2}}}+\frac{1}{\sqrt{t}}$
	no	$\mathbb{R}^m$	no	$e^{-t}+$ tr. wave
$\forall n$	yes	$\{0\}$	yes	$e^{-t}+\frac{1}{t^{\frac{m}{2}}}$
$\forall m$	no	$\cup$vs codim r	yes	conjecture : $e^{-t}+\frac{1}{t^{\frac{m}{2}}}+\frac{1}{t^{\frac{r}{(2(n-1))}}}+\frac{1}{t^{\frac{1}{2(n-1)}}}$
	no	submanifold	yes	open problem
	no	$\mathbb{R}^m$	no	open problem
$\forall n$	yes	$\{0\}$	yes	$e^{-t}+\frac{1}{t^{\frac{m}{2}}}$
$\forall m$	no	vs codim r	yes	$e^{-t}+\frac{1}{t^{\frac{m}{2}}}+\frac{1}{t^{\frac{r}{2}}}+\frac{1}{\sqrt{t}}$
$n_1=1$				

7.3 Improvements for nonlinear systems

In this section, we study non linear systems of balance laws of the form

$$\frac{\partial w}{\partial t}+\sum_{j=1}^{m}\frac{\partial F_j(w)}{\partial x_j}=Q(w) \tag{7.19}$$

where $m,n\in\mathbb{N}^*$, $w:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^n$, $w=w(t,x)$ is the unknown, $F_j,Q:\mathbb{R}^n\to\mathbb{R}^n$ are smooth functions, and

$$Q(w):=\begin{pmatrix}0\\ q(w)\end{pmatrix}$$

where $0\in\mathbb{R}^{n_1}$ and $q(w)\in\mathbb{R}^{n_2}$.

We consider a constant equilibrium $W_e\in\mathbb{R}^n$ i.e. $Q(W_e)=0$. The aim of this section is to investigate under what conditions the source term may prevent the breakdown of smooth solutions, in a neighborhood of W_e. This question has been addressed earlier, for instance, in [168], under the following assumptions **(H0)**-**(H2)** in order to deal with strictly entropy dissipative symmetrizable systems.

(H0) : The differential $d_{w_2}q(W_e)$ is invertible.

(H1) : There exists a strictly convex entropy $\eta=\eta(w)$, defined in a convex compact neighborhood G of W_e, such that $d_w^2\eta(w)d_wF_j(w)$ is symmetric for every $w\in G$ and for every $j\in\{1,...,m\}$.

(H2) : There exists a constant $C_G>0$ such that, for every $w\in G$,

$$[d_w\eta(w)-d_w\eta(W_e)]\cdot Q(w)\leqslant -C_G|Q(w)|^2,\forall w\in G. \tag{7.20}$$

Under these conditions, in [168] the existence of global smooth solutions in a neighborhood of W_e was proved, when the linearized system around W_e satisfies (SK).

We shall also assume that the hypotheses **(H0)**-**(H2)** are fulfilled. Moreover, in order to simplify the proof, we will make the following assumption **(H3)**.

(H3) : There exists $D \in \mathbb{R}^{n_2 \times n_2}$ positive definite such that

$$Q(w) := \begin{pmatrix} 0 \\ -Dw_2 \end{pmatrix}. \tag{7.21}$$

In Subsection 7.3.1, we make precise Yong's statement in [168] by giving an explicit estimate of the size of the neighborhood of W_e in which global existence holds. The possibility of measuring this size explicitly is a key ingredient in the degenerate case, studied in Subsection 7.3.2. In Subsection 7.3.2, we consider a constant degenerate equilibrium W_e and we assume the existence of a sequence of non degenerate equilibria $(W_e^p)_{p\in\mathbb{N}}$ converging to W_e and such that the quantity N_{*,W_e^p} in (7.10) measuring the decay rate (7.12) for the linearized system around W_e^p converges to zero slowly enough (this may happen for appropriate nonlinearities). Then, we prove the existence of global smooth solutions for (7.19) in a neighborhood of W_e. Our result takes advantage of the contribution of the nonlinearity, when the linearized system is degenerate. Our proof is inspired by Coron's return method for nonlinear control (see [65]). We end this section with an example of application of this theorem to relaxation problems in Subsection 7.3.3.

7.3.1 Size of the neighborhood for global existence under (SK)

In this section, we assume that (SK) holds. More precisely, it is supposed that the following hypothesis is fulfilled.

(H4) : The linearized system around W_e satisfies (SK),

$$\mathrm{Ker}(B) \cap \{\text{eigenvectors of } A_{W_e}(\omega)\} = \{0\}, \forall \omega \in S^{m-1},$$

where $A_{W_e}(\omega) := \sum_{j=1}^m \omega_j A_j(W_e)$, $A_j(W_e) := d_w F_j(W_e)$ and

$$B := \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Thus, using the same notations as in Proposition 5, we deduce that

$$N_{*,W_e} := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA_{W_e}(\omega)^k x|^2; x \in S^{n-1}, \omega \in S^{m-1}\} > 0.$$

We now introduce the compensating function

$$K_{W_e}(\omega) := \sum_{k=1}^{n-1} \epsilon^{m_k} [A_{W_e}(\omega)^{*k} B^* B A_{W_e}(\omega)^{k-1} - A_{W_e}(\omega)^{*k-1} B^* B A_{W_e}(\omega)^k]$$

with the notations of Proposition 5 (see the definition of a compensation function in [35, Remark 5]). We know that

$$\frac{B^* + B}{2} + \frac{1}{2}(K_{W_e}(\omega) A_{W_e}(\omega) + (K_{W_e}(\omega) A_{W_e}(\omega))^*) \geqslant \frac{1}{2} N_{*,W_e}. \tag{7.22}$$

We introduce

$$C_{W_e} := \max\{\|K_{W_e}(\omega)\|; \omega \in S^{m-1}\}, \tag{7.23}$$

$$\delta_{W_e} := \frac{1}{2} \min\{\frac{1}{C_{W_e}}, \frac{N_{*,W_e}}{2C_{W_e}^2 \|D\|^2 + \|D\| N_{*,W_e}}\}. \tag{7.24}$$

Theorem 28 *We assume that **(H0)**, **(H1)**, **(H2)**, **(H3)** and **(H4)** are fulfilled. Let $s \geqslant [m/2]+2$ be an integer and $\mathcal{M} > 0$ be such that*

$$\|Q\|_{C^s(G)} + \sum_{j=1}^{m} \|A_j\|_{C^s(G)} \leqslant \mathcal{M}. \tag{7.25}$$

Then, there exist $\nu_p = \nu_p(\eta, G, \mathcal{M}) > 0$ for $p = 1, 2$ such that, for every $w_0 \in W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)$ with

$$\|w_0 - W_e\|_{H^s(\mathbb{R}^m,\mathbb{R}^n)} < \nu_1 \min\{\delta_{W_e} N_{*,W_e}, \frac{N_{*,W_e}}{C_{W_e}}, 1\},$$

the system (7.19) has a unique global solution

$$w \in C^0([0,+\infty), W_e + H^s(\mathbb{R}^m, \mathbb{R}^n))$$

satisfying, for every $T > 0$,

$$\|w(T) - W_e\|_{H^s}^2 + \int_0^T \|w_2\|_{H^s}^2 + \delta_{W_e} N_{*,W_e} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 \leqslant \nu_2 \|w_0 - W_e\|_{H^s}^2. \tag{7.26}$$

We refer to [98] for examples of application of this result in 1D ($m = 1$). The proof is the same as in [168], we just pay more attention to the behavior of the constants, in the inequalities, with respect to the different parameters.

7.3.2 First application : global existence without (SK)

In this subsection, we assume the following hypothesis.

(H5) The linearized system around W_e does not satisfy (SK), but there exists a constant equilibrium $W_e^p \in G$ such that the linearized system around W_e^p satisfies (SK).

Theorem 29 *We assume that **(H0)**- **(H3)** and **(H5)** are fulfilled. Let $s \geqslant [m/2]+2$ be an integer and $\mathcal{M} > 0$ be such that (7.25) holds. There exist $\nu_k = \nu_k(\eta, G, \mathcal{M}) > 0$ for $k = 0, 1, 2$ such that, if*

$$N_*(W_e^p) > \nu_0 C_{W_e^p} |W_e^p - W_e|, \tag{7.27}$$

then, for every $w_0 \in W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)$ with

$$\|w_0 - W_e\|_{H^s(\mathbb{R}^m,\mathbb{R}^n)} < \nu_1 \min\left\{\delta_{W_e^p} N_{*,W_e^p}, \frac{N_{*,W_e^p}}{C_{W_e^p}}, 1\right\}, \tag{7.28}$$

the system (7.19) has a unique global solution $w \in C^0([0,+\infty), W_e + H^s(\mathbb{R}^m, \mathbb{R}^n))$ satisfying, for every $T > 0$,

$$\|w(T) - W_e\|_{H^s}^2 + \int_0^T \|w_2\|_{H^s}^2 + \delta_{W_e^p} N_{*,W_e^p} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 \leqslant \nu_2 \|w_0 - W_e\|_{H^s}^2. \tag{7.29}$$

Remark 14 *Obviously **(H5)** implies that $N_{*,W_e^p} \to 0$ when $W_e^p \to W_e$. The assumption (7.27) only says that this convergence is not too fast.*

This result may be more anecdotic than really useful (we do not know any physical system entering this framework). However, it highlights the crucial role that the nonlinearity may play, in the existence of global smooth solution, as emphasized in the following example.

Example of application of Theorem 29 : Let us consider (7.19) with $m = 1$, $n = 2$,

$$F(w) = \begin{pmatrix} F^{(1)}(w) \\ F^{(2)}(w) \end{pmatrix}, \quad Q(w) = \begin{pmatrix} 0 \\ w_2 \end{pmatrix}, \quad W_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad W_e^p = \begin{pmatrix} a_p \\ 0 \end{pmatrix},$$

where $a_p \neq 0$ and $a_p \to 0$. We assume

–
$$\frac{\partial F^{(2)}}{\partial w_1}(W_e) = 0,$$

–
$$\frac{\partial F^{(1)}}{\partial w_2}(w) = \frac{\partial F^{(2)}}{\partial w_1}(w), \forall w \in G, \tag{7.30}$$

where G is a neighborhood of W_e in $\mathbb{R}^2$,

– there exists $M > 0$ such that

$$\Big|\frac{\partial F^{(2)}}{\partial w_2}(W_e^p)\Big| \leqslant M\Big|\frac{\partial F^{(1)}}{\partial w_2}(W_e^p)\Big|, \forall p \in \mathbb{N}. \tag{7.31}$$

The assumption **(H0)** is fulfilled because $q(w) = -w_2$ thus $d_{w_2}q(w) = -1$ is invertible. The assumption **(H1)** holds with $\eta(w) := |w|^2$ because $d_w F(w)$ is symmetric thanks to the assumption (7.30). The assumption **(H2)** is fulfilled with $C_G = 1$. It is clear that **(H3)** holds.

By definition, $N_{*,W}$ is the smallest eigenvalue of the non negative matrix

$$\begin{pmatrix} \epsilon^{m_1}\left(\frac{\partial F^{(2)}}{\partial w_1}(W)\right)^2 & \epsilon^{m_1}\frac{\partial F^{(2)}}{\partial w_1}(W)\frac{\partial F^{(2)}}{\partial w_2}(W) \\ \epsilon^{m_1}\frac{\partial F^{(2)}}{\partial w_1}(W)\frac{\partial F^{(2)}}{\partial w_2}(W) & 1+\epsilon^{m_1}\left(\frac{\partial F^{(2)}}{\partial w_2}(W)\right)^2 \end{pmatrix},$$

thus $N_{*,W_e} = 0$, i.e. (SK) is not satisfied for the linearized system around W_e and, for $\epsilon > 0$ small enough, we have

$$N_{*,W_e^p} \geqslant \frac{1}{2}\epsilon^{m_1}\Big|\frac{\partial F^{(2)}}{\partial w_1}(W_e^p)\Big|^2.$$

By definition of $C_{W_e^p}$ and thanks to (7.31), we have

$$C_{W_e^p} \leqslant 2\epsilon^{m_1}(1+M)\Big|\frac{\partial F^{(2)}}{\partial w_1}(W_e^p)\Big|$$

Thus, (7.27) holds in particular when

$$\Big|\frac{\partial F^{(1)}}{\partial w_2}(W_e^p)\Big| > 4\nu_0(1+M)|W_e^p - W_e|,$$

which is valid, in particular, when

$$\Big|\frac{\partial^2 F^{(1)}}{\partial w_2 \partial w_1}(W_e)\Big|$$

is large enough. At this point, it is clear that the assumption (7.27) takes advantage of the nonlinearity.

7.3.3 Second application : uniform well-posedness

Let us consider a partially dissipative nonlinear hyperbolic system of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j(w_2) \frac{\partial w}{\partial x_j} = -\frac{1}{\tau} B w, \tag{7.32}$$

where B has form (7.2), $\tau > 0$ is a relaxation parameter, $w = (w_1, w_2)$ with $w_1 \in \mathbb{R}^{n_1}$ and $w_2 \in \mathbb{R}^{n_2}$ and the matrices $A_j(w_2)$ are symmetric. To simplify, we assume that

$$\Re\langle DX, X\rangle \geqslant |X|^2, \forall X \in \mathbb{C}^{n_2}.$$

Assume we are interested in the limit of the solutions w_τ of this system when the parameter τ tends to zero. Then, one needs a uniform well posedness result, with respect to the parameter τ : there exists $\delta > 0$ such that, for every $w_0 \in H^s$ with $\|w_0\|_{H^s} < \delta$ and for every $\tau \in (0,1)$, the system (7.32) has a unique global solution. Such a situation arises in article [72] where the particular case of the multidimensional isothermal Euler equation is studied.

The goal of this section is to show that this uniform well posedness can be proved thanks to the explicit size of the neighborhood for global existence proved in Section 7.3.1. Precisely, we prove the following result.

Theorem 30 *We assume **(H0)**, **(H1)**, **(H2)**, **(H3)** and **(H4)** with $W_e = 0$. Let $s \geqslant [m/2] + 2$ be an integer. Then, there exists $\nu_p = \nu_p(\eta, G, A_1, ...A_m, B) > 0$ for $p = 1, 2$ such that, for every $w_0 \in H^s(\mathbb{R}^m, \mathbb{R}^n)$ with $\|w_0\|_{H^s} \leqslant \nu_1$ and for every $\tau \in (0,1)$, the system (7.32) has a unique global solution $w \in C^0([0, +\infty), H^s(\mathbb{R}^m, \mathbb{R}^n))$ satisfying, for every $T > 0$,*

$$\|w(T)\|_{H^s}^2 + \frac{1}{\tau}\int_0^T \|w_2(t)\|_{H^s}^2 dt + \tau \int_0^T \|\nabla w\|_{H^{s-1}}^2 dt \leqslant \nu_2 \|w_0\|_{H^s}^2. \tag{7.33}$$

The proof of Theorem 30 follows the one of Theorem 28, we only precise the behavior of the different constants with respect to the parameter τ. It also follows the one of [72]. Notice that the dependence of the matrices A_j with respect to w_2 (and not w) is crucial in the proof. : with $A_j = A_j(w)$ the same result is an open question.

7.4 Conclusion, open problems, perspectives

7.4.1 Complete classification of the asymptotic behavior of linear systems

As emphasized in the table of Section 7.2.3, the complete classification of the asymptotic behavior for the linear partially dissipative hyperbolic systems is an open problem. In particular, the decomposition of the solutions of (7.1) when $m \geqslant 2$, $n \geqslant 3$ are arbitrary is not known in general.

The simplest situation not covered in this article should be the case when the set of degeneracy is a finite union of vector subspaces of $\mathbb{R}^m$. We conjecture that, in that case, the following inequality holds

$$N_{*,\epsilon}(\omega) \geqslant c\mathrm{dist}(\omega, \mathcal{D})^{2(n-1)}. \tag{7.34}$$

The intuition of the exponent $2(n-1)$ comes from the definition

$$N_{*,\epsilon}(\omega) := \min \left\{ \sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2 ; x \in S^{n-1} \right\}.$$

Indeed, for every $x \in S^{n-1}$, $\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2$ is a polynomial of ω with degree $\leqslant 2(n-1)$. If the conjecture (7.34) is valid, then Proposition 7 leads to a decomposition for the solutions of (7.1) when $\mathcal{D}(A_1, ..., A_m, B)$ is a finite union of vectors subspaces of $\mathbb{R}^m$. This decomposition is written in the seventh line of the array of Section 7.2.3.

When the set of degeneracy is a strict algebraic submanifold of $\mathbb{R}^m$, the situation is more difficult because it is not sufficient to understand the behavior of $N_{*,\epsilon}(\omega)$, one also needs the analogue of Proposition 7. A parameterization of the submanifold is required to use the same proof strategy.

Another open problem is the nature of the non dissipated solutions and the decomposition of the solutions when $\mathcal{D}(A_1, ..., A_m, B) = \mathbb{R}^m$.

7.4.2 Nonlinear systems

The analogue of Theorem 30 is not known when the matrices A_j depend on w (and not only on its second component w_2).

7.4.3 Lyapunov functions

In Section 7.2, we propose a strict Lyapunov function for the ODE (7.11). One may deduce from this result a strict Lyapunov function for the PDE (7.1), when $\mathcal{D}(A_1, ..., A_m, B) \neq \mathbb{R}^m$,

$$\mathcal{L}(t) := \int_{\mathbb{R}^m} |\hat{w}(t,\xi)|^2 + \min\left\{ |\xi|, \frac{1}{|\xi|} \right\} \sum_{k=1}^{n-1} \epsilon^{m_k} \Im \left\langle BA\left(\frac{\xi}{|\xi|}\right)^{k-1} \hat{w}(t,\xi), BA\left(\frac{\xi}{|\xi|}\right)^{k} \hat{w}(t,\xi) \right\rangle d\xi$$

Indeed, it satisfies

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -c \int_{\mathbb{R}^m} \min\{1, |\xi|^2\} N_{*,\epsilon}\left(\frac{\xi}{|\xi|}\right) |\hat{w}(t,\xi)|^2 d\xi.$$

This Lyapunov function relies on the Fourier transform. It would be interesting to avoid this use of the Fourier transform, in order to treat similarly linear hyperbolic systems with variable coefficients. Such a Lyapunov function may be useful for feedback stabilization problems, as, for example, in [68].

By slightly adapting the strategy presented in this Chapter, one may treat such cases, under quite restrictive assumptions. First, let us consider, the simpler constant coefficient hyperbolic system

$$\begin{cases} \frac{dw}{dt} + A\frac{dw}{dx} = -Bw, x \in \mathbb{T}, \\ \int_{\mathbb{T}} w(t,x)dx = 0, \end{cases}$$

where $w : (t,x) \in \mathbb{R} \times \mathbb{T} \mapsto \mathbb{R}^m$, the matrix A is symmetric and B has form (7.2). Then the following expression defines a strict Lyapunov function

$$\widetilde{\mathcal{L}}(t) := \int_{\mathbb{T}} \left[|w(t,x)|^2 + \sum_{k=1}^{n-1} \epsilon^{m_k} \left\langle BA^{k-1} \frac{\partial^{k-1} w}{\partial x^{k-1}}(t,x), BA^k \frac{\partial^k w}{\partial x^k}(t,x) \right\rangle \right] dx. \tag{7.35}$$

Indeed, with the same arguments as in the proof of Proposition 5, we get the following inequality

$$\frac{d}{dt}\widetilde{\mathcal{L}}(t) \leqslant -c \int_{\mathbb{T}} \left(|Bw(t,x)|^2 + \sum_{k=1}^{n-1} \epsilon^{m_k} |BA^k \partial_x^k w(t,x)|^2 \right) dx,$$

which proves the exponential convergence to zero in $L^2(\mathbb{T})$. It is interesting to compare this Lyapunov function with the one used in [68], in which the authors also use higher order terms, but rather to deal with the nonlinearity.

Now, let us assume that the matrix B has variable coefficients. Then, one may still prove that (7.35) defines a strict Lyapunov function, under some appropriate conditions. For instance, let us assume that A is constant and $B = B(x)$ depends on x, but in the following way

$$B(x) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} + D(x) \end{pmatrix}, \tag{7.36}$$

where $D \in \mathbb{R}^{n_2 \times n_2}$ is small :

$$X^t(I + D(x))X \geqslant \frac{1}{2}\|X\|^2, \forall X \in \mathbb{R}^{n_2} - \{0\}, \forall x \in \mathbb{T}.$$

Then, similar computations prove that $\widetilde{\mathcal{L}}(t)$ is still a strict Lyapunov function.

However, when the set dissipated by the matrix $B = B(x)$ depends on the point x, then new ideas need to be introduced.

Chapitre 8

Controllability of the Kolmogorov equation (A8) [34]

8.1 Introduction

Equation : The article **(A8)** [34] deals with the Kolmogorov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial v^2} = u(t,x,v)1_\omega(x,v), (x,v) \in \Omega, t \in (0,+\infty), \tag{8.1}$$

where Ω is an open subset of $\mathbb{R}^2$, $\omega \subset \Omega$, 1_ω is the characteristic function of this set and $u(t,x,v)$ is a source term located on the subdomain ω. It is a linear control system in which

- the state is f,
- the control is u and it is supported in the subset ω.

We investigate the null controllability of this system.

The heat equation : In order to motivate this study, let us recall few known results concerning the heat equation. The null and approximate controllability of the heat equation are essentially well understood subjects both for linear equations and for semilinear ones, both for bounded and unbounded domains (see for instance [76, 82, 84, 85, 86, 96, 141, 90, 126, 130, 115, 78, 79]). Let us focus on the linear heat equation

$$\begin{cases} y_t(t,x) - \Delta y(t,x) = u(t,x)1_\omega(x), x \in \Omega, t \in (0,T), \\ y = 0 \text{ on } (0,T) \times \partial\Omega, \\ y(0) = y^0, \end{cases} \tag{8.2}$$

where Ω is an open subset of $\mathbb{R}^l$, $l \in \mathbb{N}^*$ and ω a subset of Ω. One has the following theorem, which is due to H. Fattorini and D. Russell [83, Theorem 3.3] if $l = 1$, to O. Imanuvilov [139, 140] (see also the book [88] by A. Fursikov and O.Imanuvilov), and to G. Lebeau and L. Robbiano [90] for $l \geqslant 2$. We also refer to the book [65, Theorem 2.66] by J.-M. Coron for a pedagogical presentation.

Theorem 31 *Let us assume that Ω is bounded, of class C^2 and connected, $T > 0$, and ω is a non empty open subset of Ω. Then the control system (8.2) is null controllable in time T : for every $y^0 \in L^2(\Omega,\mathbb{R})$, there exists $u \in L^2((0,T)\times\Omega,\mathbb{R})$ such that the solution of (8.2) satisfies $y(T) = 0$.*

In particular, the heat equation on a bounded domain is null controllable

– in arbitrarily small time,
– with an arbitrarily small control support ω.

As a consequence of Theorem 31, we also have the following result [163].

Theorem 32 *Let us assume that $\Omega = \mathbb{R}^l$, $T > 0$, and ω is the complementary in $\mathbb{R}^l$ of a compact set. Then the control system (8.2) is null controllable in time T : for every $y^0 \in L^2(\mathbb{R}^l, \mathbb{R})$, there exists $u \in L^2((0,T)\times\mathbb{R}^l, \mathbb{R})$ such that the solution of (8.2) satisfies $y(T) = 0$.*

In particular, the heat equation on the whole space is null controllable

– in arbitrarily small time,
– when the control support is the complementary of a compact subset of $\mathbb{R}^l$.

Open problem for the Kolmogorov equation : The Kolmogorov equation (8.1) diffuses both in space and velocity variables : it diffuses in v thanks to $\partial_v^2 f$ and also in x, in a hidden way, thanks to an interplay between the transport term $v\partial_x f$ and the diffusive term $\partial_v^2 f$ (see, for instance, [166] for the study of the asymptotic behavior, see also Lemmas 1 and 2 of the article **(A8)** [34]).

A convincing heuristic consists in considering formally the change of variable and function

$$f(t,x,v) = h(t, x - vt, v)$$

which leads to the equation

$$\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial v^2} + 2t\frac{\partial^2 h}{\partial x \partial v} - t^2\frac{\partial^2 h}{\partial x^2} = 0.$$

In this formulation, it is clear that the Kolmogorov equation is a degenerate parabolic equation (or a degenerate coercive system) : the quadratic form $v^2 - 2txv + t^2x^2 = (v - tx)^2$ is non negative, but it is not positive definite ; moreover, the degenerate direction changes along time. Actually, the Kolmogorov equation is one of the simplest hypocoercive system [166]. Thus, it is natural to ask if the null controllability results known for the heat equation also hold for the Kolmogorov equation.

Bibliography about related works : Degenerate parabolic equations are also studied in [49, 48, 47]. The authors consider parabolic operators that degenerate on the boundary of the space domain, for instance

$$w_t + (x^\alpha w_x)_x = f 1_\omega, x \in (0,1).$$

They prove a Carleman estimate for the solutions of the adjoint system. The proof is based on the choice of suitable weighted functions and Hardy type inequalities. For the Kolmogorov equation (8.1) the situation is a bit different because the operator degenerates everywhere (in the spatial domain), but along a direction that changes along time.

Finally, let us mention the reference [128], in which a simplified version of the Kolmogorov equation (the linearized Crocco type equation) is studied. This equation mixes transport in the variable x and diffusion in the variable v but in a simpler way than the Kolmogorov equation, because the transport in variable x is done at constant velocity 1 instead of velocity v,

$$\begin{cases} f_t + f_x - f_{vv} = u(t,x,v)1_\omega(x,v), (t,x,v) \in (0,T) \times (0,L) \times (0,1), \\ f(t,x,0) = f(t,x,1) = 0, \\ f(t,0,v) = f(t,L,v). \end{cases}$$

Because of this decoupling of the transport and the diffusion phenomena, the linearized Crocco type equation does not diffuse in variable x, thus the problem of using an arbitrarily small control domain becomes very different.

For a given open subset ω of $\Omega := (0,L) \times (0,1)$, the authors of [128] prove the property of "regional null controllability ", which consists in controlling the solution within the domain of influence of the controls located in ω.

However, for the Kolmogorov equation, the result may be different, because, as we said above, this equation diffuses both in variables v and x, thus the domain of influence of an arbitrarily small subset ω may be the whole domain Ω in any time $T > 0$. This problem is still open.

8.2 Results

We investigate the null controllability of the equation (8.1) in two different geometric configurations,

$$\Omega_1 = \mathbb{R}_x \times \mathbb{R}_v \ , \ \omega_1 = \mathbb{R}_x \times [\mathbb{R} - (a_1, b_1)]_v$$

where $-\infty < a_1 < b_1 < +\infty$ and

$$\Omega_2 = (0, 2\pi)_x \times (0, 2\pi)_v \ , \ \omega_2 = (0, 2\pi)_x \times (a_2, b_2)_v,$$

where $0 \leqslant a_2 < b_2 \leqslant 2\pi$. More precisely, we study the Cauchy problems

$$\begin{cases} \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial v^2} = u(t,x,v)1_{\omega_1}(v) \ , \ (x,v) \in \Omega_1 \ , \ t \in (0,+\infty), \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{8.3}$$

and

$$\begin{cases} \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial v^2} = u(t,x,v)1_{\omega_2}(x,v) \ , \ (x,v) \in \Omega_2 \ , \ t \in (0,+\infty), \\ f(t,0,v) = f(t,2\pi,v), \\ f(t,x,0) = f(t,x+2\pi t,2\pi), \\ \partial_v f(t,x,0) = \partial_v f(t,x+2\pi t,2\pi), \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{8.4}$$

The boundary conditions in (8.4) may seem strange. We chose them to ensure that the function $h(t,x,v) := f(t,x+vt,v)$ is 2π-periodic with respect to x and v, which facilitates the Fourier analysis of solutions. Notice that, thanks to the second line of (8.4), one can identify the function f and the function from $(0,+\infty)_t \times \mathbb{R}_x \times (0,2\pi)_v$ to $\mathbb{R}$, which is 2π-periodic with respect to the variable x and coincides with f on $(0,+\infty)_t \times (0,2\pi)_x \times (0,2\pi)_v$. This gives sense to the third and fourth lines of (8.4).

The main result of this article guarantees the null controllability of systems (8.3) and (8.4).

Theorem 33 *For every $T > 0$ and $f_0 \in L^2(\Omega_1, \mathbb{R})$ (resp. $f_0 \in L^2(\Omega_2, \mathbb{R})$), there exists $u \in L^2((0,T) \times \Omega_1, \mathbb{R})$ (resp. $u \in L^2((0,T) \times \Omega_2, \mathbb{R})$) such that the solution of (8.3) (resp. (8.4)) satisfies $f(T) = 0$.*

By duality, this result is equivalent to the following observability inequalities for the corresponding adjoint systems (see for instance [65, Lemma 2.48]).

Theorem 34 *For every $T > 0$, there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega_1, \mathbb{R})$, the solution of*

$$\left\{ \begin{array}{l} \frac{\partial g}{\partial t} - v\frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial v^2} = 0, (x,v) \in \Omega_1, t \in (0,T), \\ g(0,x,v) = g_0(x,v), (x,v) \in \Omega_1 \end{array} \right. \tag{8.5}$$

satisfies

$$\int_{\Omega_1} |g(T,x,v)|^2 dxdv \leqslant C \int_0^T \int_{\omega_1} |g(t,x,v)|^2 dxdvdt.$$

Theorem 35 *For every $T > 0$, there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega_2, \mathbb{R})$, the solution of*

$$\left\{ \begin{array}{l} \frac{\partial g}{\partial t} - v\frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial v^2} = 0, (x,v) \in \Omega_2, t \in (0,T), \\ g(t,0,v) = g(t,2\pi,v), \\ g(t,x,0) = g(t,x+2\pi(T-t),2\pi), \\ \frac{\partial g}{\partial v}(t,x,0) = \frac{\partial g}{\partial v}(t,x+2\pi(T-t),2\pi), \\ g(0,x,v) = g_0(x,v), \end{array} \right. \tag{8.6}$$

satisfies

$$\int_{\Omega_2} |g(T,x,v)|^2 dxdv \leqslant C \int_0^T \int_{\omega_2} |g(t,x,v)|^2 dxdvdt.$$

8.3 Technics

The strategy relies on the use of the Fourier transform (continuous Fourier transform on Ω_1, Fourier series on Ω_2). The two cases are similar, thus, let us only present the strategy on Ω_1.

Let us apply the Fourier transform, in the x variable to the equation (2.11) : for every $\xi \in \mathbb{R}$, we get

$$\frac{\partial \hat{f}}{\partial t}(t,\xi,v) + i\xi v \hat{f}(t,\xi,v) - \frac{\partial^2 \hat{f}}{\partial v^2}(t,\xi,v) = \hat{u}(t,\xi,v) 1_{\mathbb{R}-[a_1,b_1]}(v), v \in \mathbb{R}^2. \tag{8.7}$$

Thus, for every $\xi \in \mathbb{R}$, the function $(t,v) \mapsto \hat{f}(t,\xi,v)$ solves a heat equation with potential $i\xi v$. Our proof uses two key ingredients, as in [105] :

- an explicit exponential decay rate for the solution of the system (8.7) with $u = 0$,

  $$\|\hat{f}(t,\xi,.)\|_{L^2(\mathbb{R}_v)} \leqslant \|\hat{f}_0(\xi,.)\|_{L^2(\mathbb{R}_v)} e^{-\xi^2 t^3/12}, \forall \xi \in \mathbb{R}, \forall t \in [0,+\infty),$$

  got by computing explicitly the Fourier transform (in x and v of $f(t)$).
- an explicit cost (in ξ) for the null controllability of the heat equation (8.7), of the form $e^{C(T)\max\{1,\sqrt{|\xi|}\}}$, proved thanks to a new Carleman estimate.

Now, the proof of the null controllability is in two steps : given a time $T > 0$

- on the time interval $[0, T/2]$, we do not put any control ($u = 0$), in order to take advantage of the natural dissipation of the system

  $$\|\hat{f}(T/2,\xi,.)\|_{L^2(\mathbb{R}_v)} \leqslant e^{-\xi^2 T^3/96}, \forall \xi \in \mathbb{R},$$

– on the time interval $[T/2, T]$, we use the control $u(t,x,v)$ such that $\hat{u}(t,\xi,v)$ realizes the null controllability of $\hat{f}(T/2,\xi,v)$, for every $\xi \in \mathbb{R}$.

Then, one has to check that the control we get indeed belongs to $L^2((0,T)\times\mathbb{R}^2)$. This is a consequence of the previous result :

$$\begin{aligned}
\int_{T/2}^{T}\int_{\mathbb{R}^2}|u(t,x,v)|^2 dxdvdt &= \int_{T/2}^{T}\int_{\mathbb{R}^2}|\hat{u}(t,\xi,v)|^2 d\xi dvdt \\
&\leqslant \int_{T/2}^{T}\int_{\mathbb{R}} e^{C(T)\max\{1,\sqrt{|\xi|}\}}\|\hat{f}(T/2,\xi,.)\|^2_{L^2(\mathbb{R}_v)} d\xi dt \\
&\leqslant \int_{T/2}^{T}\int_{\mathbb{R}} e^{C(T)\max\{1,\sqrt{|\xi|}\}} e^{-\xi^2 T^3/48}\|\hat{f}_0(\xi,.)\|^2_{L^2(\mathbb{R}_v)} d\xi dt \\
&\leqslant C\|f_0\|^2_{L^2(\mathbb{R}^2)} < +\infty.
\end{aligned}$$

8.4 Conclusion, open problems, perspectives

This article is a first step in the understanding of the null controllability of hypocoercive systems : we propose a way to adapt the classical arguments (Carleman estimates) in order to deal with a possible hypocoercivity. However, this result is not completely satisfying, because the geometrical configurations are very restrictive. In particular, the control locations are invariant under translation along the x axis, because of the use of the Fourier transform.

Chapitre 9

Control and micromagnetism (A7,P1) [10, 11]

9.1 Introduction

The study of small magnetic particles has become a very important topic, in particular for the development of technological devices such as those used for magnetic recording. In this field, switching the magnetization inside the magnetic sample is of particular relevance.

In the references **(A7,P1)** [10, 11], we address the question of studying mathematically the possibility of switching the magnetization inside an elongated particle with external magnetic fields that are uniform in space (but may be variable in time). As we shall see, although we will restrict to small ellipsoidal ferromagnetic particles, we will consider the full PDE problem, and both weak and strong solutions.

The magnetization m inside a ferromagnetic body, located in a space domain Ω, is a three dimensional vector field, defined on Ω and constrained to be of constant magnitude through the sample. After a suitable renormalization, we consider this magnitude to be equal to 1. The evolution of the magnetization inside a ferromagnetic body is modelized by the Landau-Lifschitz equation,

$$\tfrac{\partial m}{\partial t} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m), \text{ in } \Omega. \tag{9.1}$$

Here, $H(m)$ is the total magnetic field induced by several physical phenomena (exchange, stray-field, anisotropy, exterior field), $\alpha > 0$ is a damping coefficient which depends on the material (we refer the reader to [45] or [103] for a more complete description of the physical model). In this equation, $\langle ., . \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^3$ and $\wedge$ is the vectorial product. Equivalently, at least for smooth solutions, the equation (9.1) may be written under the so-called Gilbert form

$$\frac{\partial m}{\partial t} - \alpha \left(m \wedge \frac{\partial m}{\partial t} \right) = -(1+\alpha^2)\left(m \wedge H(m)\right), \tag{9.2}$$

and under the form,

$$\alpha \frac{\partial m}{\partial t} + \left(m \wedge \frac{\partial m}{\partial t} \right) = (1+\alpha^2)[H(m) - \langle H(m), m\rangle m]. \tag{9.3}$$

For a ferromagnetic body without anisotropy, the magnetic field $H(m)$ can be expressed, in order to emphasize the dependence on the (non-constant in time) external magnetic field, as

$$H(m) = -\frac{\partial \mathcal{E}}{\partial m} + H_{ext},$$

where $\mathcal{E}(m)$ is the micromagnetic energy associated to a given magnetization m,

$$\mathcal{E}(m) := \frac{A}{2}\int_\Omega |\nabla m|^2 - \frac{1}{2}\int_\Omega \langle H_d(m), m\rangle. \tag{9.4}$$

This leads to

$$H(m) := A\Delta m + H_d(m) + H_{ext}, \tag{9.5}$$

where H_{ext} is the uniform in space external magnetic field applied to the sample, A is the so-called exchange constant [45], that will be taken equal to one to simplify the presentation and $H_d(m)$ is the stray field generated by the magnetization m itself via the following dimensionless Maxwell equations

$$\begin{cases} H_d(m) = \nabla\phi, \text{ in } \mathbb{R}^3, \\ \Delta\phi = -\text{div}(\overline{m}), \text{ in } \mathbb{R}^3, \\ H_d(m) \text{ vanishes at infinity,} \end{cases} \tag{9.6}$$

where

$$\overline{m} = \begin{cases} m \text{ in } \Omega, \\ 0 \text{ in } \mathbb{R}^3 \setminus \Omega. \end{cases} \tag{9.7}$$

The natural boundary conditions are of Neumann type, thus, we will work on the following Cauchy problem

$$\begin{cases} \dfrac{\partial m}{\partial t} = \alpha[H(m) - \langle H(m), m\rangle m] - m \wedge H(m), \ x \in \Omega, \ t \in (0,T) \\ \dfrac{\partial m}{\partial \nu}(t,x) = 0, \ x \in \partial\Omega, \ t \in (0,T), \\ m(0,x) = m_0(x), \ x \in \Omega. \end{cases} \tag{9.8}$$

It is a non linear control system in which

- the state is the magnetization m, with $m(t) : \Omega \to S^2$, for every t,
- the control is the external magnetic field $H_{ext} : t \in \mathbb{R}_+ \mapsto \mathbb{R}^3$.

In the articles **(A7,P1)** [10, 11], we are interested in the existence and the properties of a such control H_{ext} that steers m from $m(0) = u$ to $m(T) = -u$ where u and $-u$ are global minimizers of the micromagetic energy $\mathcal{E}$. We will also give a couple of results in the problem similar to (9.8) but posed in 2D. By this, we mean that the domain is bidimensional, but the magnetization still takes values into S^2. However, the stray field satisfies (9.6) and (9.7) but in $\mathbb{R}^2$. This models an infinite ferromagnetic cylindrical rod along the axis of which the solution is invariant. As a consequence of (9.6) and (9.7) the component of the stray field parallel to the axis of the cylinder vanishes.

Finally, let us also quote the paper by Carbou, Labbé and Trélat [51] which also treats a control problem in micromagnetics, but in the completely different context of moving a wall in a nano-wire, and the paper by Labbé, Privat and Trelat [116] dealing with the stability of steady-states for a network of ferromagnetic nanowires.

This chapter is organized as follows. In Section 9.2, we consider a ferromagnetic body having an ellipsoidal shape. Then, the stray field of uniform magnetizations is uniform, thus, a subclass of solutions of (9.8) are uniform magnetizations that solve an ordinary differential equation (ODE). We study the switching on these uniform magnetizations under different restrictions on the external magnetic field, motivated by the applications :

– in a first step, we do not impose any constraint on $H_{ext} : t \in \mathbb{R} \mapsto \mathbb{R}^3$,
– in a second step, we ask H_{ext} to take values in a vector subspace V of $\mathbb{R}^3$ of dimension 2, as it is the case in MRAMs (Magnetic Random Access Memories),
– in a third step, we study the spin induced switching, for which H_{ext} is of the form

$$H_{ext}(t,x) = h(t)m(t,x) \wedge e, \tag{9.9}$$

where $h(t) \in \mathbb{R}$ is an amplitude and $e \in S^2$ is fixed.

The Section 9.3 is dedicated to weak solutions for the partial differential equation (PDE) (9.8). In Section 9.3.1, we prove the existence of weak solutions of (9.8). In Section 9.3.2, we study their convergence to uniform magnetizations when the size of the domain Ω goes to zero. This already shows that the external field found in Section 9.2 allows an 'approximate' (in an unusual sense) switching on any sufficiently small domain in the very general sense of weak solutions. To go further, we need more regularity and strong solutions. This imposes restrictions on either the shape of the domain or the regularity and smallness of the initial condition. Namely, Section 9.4 is dedicated to smooth solutions of (9.8). First, we prove the existence and uniqueness of local (in time) smooth solutions when Ω is a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then, we prove that such local solutions indeed provide global solutions when Ω is a 2D bounded domain and when the initial condition is in a H^1-neighborhood of constant magnetizations. Finally, we prove the existence of global smooth solutions when Ω is a small 3D ellipsoid domain and when the initial condition is in a H^2-neighborhood of constant magnetizations. Contrarily to the preceding results where we follow the strategy developed by [50], the latter result involves different ideas.

In Section 9.5, we work with small 2D or 3D ellipsoid domains Ω. We propose explicit external fields that exponentially stabilize the uniform stationary solutions.

In Section 9.6, we propose a way to realize the approximate switching of PDE solutions on small 2D or 3D ellipsoidal domains.

9.2 A Landau-Lifschitz ODE

It is well known that, when Ω is a 3D ellipsoidal domain over which the magnetization is constant, the stray field is also constant on Ω and therefore satisfies

$$H_d(m) = -Dm \text{ on } \Omega$$

where D is a symmetric positive matrix. Up to an orthonormal change of coordinates, we may take

$$D := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$

where $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant 1$ depend on the size of the three axis of the ellipsoid.

In this case, the Landau-Lifschitz equation becomes the ordinary differential system

$$\begin{cases} \dfrac{dm}{dt} = \alpha[H_0(m) - \langle H_0(m), m\rangle m] - m \wedge H_0(m), \\ m(0) = m_0, \\ m : \mathbb{R}_+ \to S^2. \end{cases} \tag{9.10}$$

where

$$H_0(m) = -Dm + H_{ext}. \tag{9.11}$$

3D external fields

Viewing H_{ext} as a control parameter, (9.10) turns out to be a flat system : for every reference path $m_{ref} \in H^1((0,T), S^2)$ (for some $T > 0$), there exists an external field $H_{ext}[m_{ref}(t)] \in L^2((0,T), \mathbb{R}^3)$ such that the unique solution m of (9.10) with initial condition $m_0 = m_{ref}(0)$, and external field $H_{ext}(m_{ref})$ coincides with m_{ref}

$$m(t) = m_{ref}(t), \forall t \in (0,T).$$

Indeed, (9.10) rewrites as

$$\alpha \frac{dm}{dt} + m \wedge \frac{dm}{dt} = (1+\alpha^2)[-Dm + H_{ext} + \langle Dm - H_{ext}, m\rangle m], \tag{9.12}$$

thus, for every path $m \in H^1((0,T), S^2)$, the magnetic field

$$H_{ext}(m) := \frac{1}{1+\alpha^2}\left\{\alpha \frac{dm}{dt} + m \wedge \frac{dm}{dt}\right\} + Dm - \langle Dm, m\rangle m \tag{9.13}$$

belongs to $L^2((0,T), \mathbb{R}^3)$ and, since $\langle H_{ext}(m), m\rangle = 0$, it allows to follow m.

Thanks to the explicit expression (9.13), we study several qualitative questions : existence, uniqueness and profile of optimal controls, behavior of the optimal control cost as $T \to +\infty$.

2D external fields

In this section, we study the ODE (9.10)-(9.11) under the constraint

$$H_{ext}(t) \in V, \forall t \in [0, +\infty), \tag{9.14}$$

where V is a fixed 2D vector subspace of $\mathbb{R}^3$.

First, let us introduce few notations. For $m \in S^2$, $S(m)$ is the 3×3 skew-symmetric matrix such that $S(m)v = m \wedge v$ for every $v \in \mathbb{R}^3$ and Π_m is the orthogonal projection from $\mathbb{R}^3$ to $m^\perp$,

$$\Pi_m(v) := v - \langle v, m\rangle m, \forall v \in \mathbb{R}^3, \forall m \in S^2.$$

Let $\beta := 1 + \alpha^2$. Then, the equation (9.10) may be written as

$$\frac{dm}{dt} = \beta(\alpha Id + S(m))^{-1}(\Pi_m(Dm) + \Pi_m(H_{ext})).$$

In a first step, in **(P1)**[11], we characterize the feasible trajectories (see [11] for a precise definition).

Proposition 8 *Let V be a 2D vector subspace of $\mathbb{R}^3$. Any C^1 curve $m : [0,T] \to S^2$ such that, for every $t^* \in [0,T]$ for which $m(t^*) \in V$, then $\frac{dm}{dt}(t^*)$ is transversal to V and belongs to the 1D affine subspace of $\mathbb{R}^3$*

$$\frac{dm}{dt}(t^*) \in (\alpha Id + S(m))^{-1}(\Pi_m(Dm) + V \cap m^\perp)$$

is feasible for (9.10)-(9.11)-(9.14). Thus, (9.10)-(9.11)-(9.14) is controllable.

Then, focusing on the particular case $V = \text{Span}(e_1, e_2)$, which is the most interesting for the applications to MRAMs, we prove that the magnetization switching may be done in arbitrarily small time (with large controls), or with quite small controls (in large time).

Spin induced switching

In this section, we study the ODE (9.10)-(9.11) under the constraint

$$H_{ext}(t) = h(t)m(t) \wedge e \tag{9.15}$$

where $h : [0, T] \to \mathbb{R}$ is an amplitude (the control) and $e \in S^2$ is a fixed vector. In **(P1)** [11], we prove the following result.

Proposition 9 *The system (9.10)-(9.11)-(9.15) is controllable if and only if e is not an eigenvector of D.*

The proof of this result relies on a study of the phase portrait of the system with constant controls.

9.3 PDE weak solutions

9.3.1 Existence of 3D global weak solutions

Weak solutions for Landau-Lifschitz equations have been proven to exist in [12] and [167] although either without a possibly variable in time external field or without the stray-field. We here follow the strategy of [12], and perform necessary adaptations to our case.

Definition 12 *Let $m_0 \in H^1(\Omega, S^2)$ and $H_{ext} \in L^1_{loc}(\mathbb{R}, \mathbb{R}^3)$. A function m is a weak solution of (9.8) if*

- *for every $T > 0$, $m \in H^1(Q_T, S^2)$,*
- *for every $T > 0$, for every $\Phi \in H^1(Q_T, \mathbb{R}^3)$, there holds*

$$\begin{aligned} &\int_{Q_T} \langle \tfrac{\partial m}{\partial t}, \Phi \rangle - \alpha \langle m \wedge \tfrac{\partial m}{\partial t}, \Phi \rangle dxdt \\ &= -(1+\alpha^2) \int_{Q_T} - \sum_{j=1}^{3} \langle m \wedge \tfrac{\partial m}{\partial x_j}, \tfrac{\partial \Phi}{\partial x_j} \rangle + \langle m \wedge (H_d(m) + H_{ext}), \Phi \rangle dxdt, \end{aligned} \tag{9.16}$$

- *$m(0, x) = m_0(x)$ in the trace sense,*
- *for almost every $T > 0$,*

$$\mathcal{E}(m(T)) + \frac{\alpha}{1+\alpha^2} \int_0^T \left\| \frac{\partial m}{\partial t}(t) \right\|^2_{L^2(\Omega)} dt \leqslant \mathcal{E}(m_0) + \int_0^T \int_\Omega \left\langle H_{ext}, \frac{\partial m}{\partial t} \right\rangle, \tag{9.17}$$

where $\mathcal{E}(m)$ is the micromagnetic energy defined by

$$\mathcal{E}(m) := \frac{1}{2} \int_\Omega |\nabla m|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |H_d(m)|^2$$

In [10], we prove the following result.

Theorem 36 *Let $m_0 \in H^1(\Omega, S^2)$ and $H_{ext} \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^3)$. There exists a weak solution of (9.8).*

The proof relies on Galerkin approximations.

9.3.2 Convergence of weak solutions to ODE solutions when the size of the domain goes to zero

Let Ω be a bounded open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ such that $|\Omega| = 1$. In this section, we consider the weak solutions of the Landau-Lifschitz PDE on the domain $\Omega_\lambda := \sqrt{\lambda}\Omega$, when $\lambda \to 0$, $\lambda > 0$. A change of space and time variables shows that it is equivalent to study the weak solutions of the following Landau-Lifschitz PDE on the fixed domain Ω ,

$$\begin{cases} \dfrac{\partial m}{\partial t} = \alpha[H_\lambda(m) - \langle H_\lambda(m), m\rangle m] - m \wedge H_\lambda(m), x \in \Omega, t \in (0,T), \\ \dfrac{\partial m}{\partial \nu}(t,x) = 0, x \in \partial\Omega, t \in (0,T), \\ m(0,x) = m_0(x), x \in \Omega, \end{cases} \tag{9.18}$$

with an effective magnetic field

$$H_\lambda(m) := \frac{\Delta m}{\lambda} + H_d(m) + H_{ext} \tag{9.19}$$

associated to the micromagnetic energy

$$\mathcal{E}_\lambda(m) := \int_\Omega \frac{1}{2\lambda}|\nabla m|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |H_d(m)|^2. \tag{9.20}$$

When the domain is small ($\lambda << 1$), non constant in space magnetizations are penalized. Therefore it is expected that solutions of (9.18), (9.19) should tend to the solutions of the ODE (9.10) with D defined by

$$D\widetilde{m} := -\frac{1}{|\Omega|}\int_\Omega H_d(\widetilde{m}\chi_\Omega) \ , \forall \widetilde{m} \in S^2,$$

where χ_Ω is the characteristic function of Ω. This is precisely the purpose of this section. We also quote the paper by DeSimone [75] in which the same kind of result is shown but for static problems, using $\Gamma-$convergence theory.

The convergence result proved in this section shows that the external magnetic field found in Section 9.2 allows an approximate switching for the PDE solutions, on any sufficiently small domain, in the very general sense of weak solutions.

Proposition 10 *Let Ω be a bounded open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, $\alpha > 0$, $H_{ext} \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^3)$, $\overline{m} \in S^2$. Let $(m_{\lambda 0})_{\lambda>0}$ be a sequence of $H^2(\Omega, S^2)$ such that $\dfrac{\partial m_{\lambda 0}}{\partial \nu} \equiv 0$ on $\partial\Omega$ for every $\lambda > 0$,*

$$m_{\lambda 0 \sharp} \to \overline{m} \text{ and } \int_\Omega |\nabla m_{\lambda 0}|^2 = o\left(\sqrt{\lambda}\right) \text{ when } \lambda \to 0. \tag{9.21}$$

Let m_λ be a weak solution of (9.18) such that $m_\lambda(0) = m_{\lambda 0}$ and m_{ref} be the solution of (9.10) with initial data $m_0 = \overline{m}$. Then, for every $T > 0$,

$$\|m_\lambda - m_{ref}\|_{C^0([0,T],H^1(\Omega))} \to 0 \text{ when } \lambda \to 0.$$

Remark 15 *The assumption (9.21) is not restrictive as the minimizers of $\mathcal{E}_\lambda$ do satisfy it.*

For the proof, an energy method gives bounds on ∇m_λ and $\dfrac{\partial m_\lambda}{\partial t}$. Then, we study the mean value of m_λ and we conclude with Poincaré formula.

It is well known (see [12] for examples) that, for $\lambda > 0$ fixed, there may not be uniqueness for the weak solutions of (9.18), (9.19). However, all these weak solutions converge in $C^0([0,T], H^1(\Omega))$ to the same function m_{ref}, when $\lambda \to 0$.

Although restricted to approximate controllability type results, the preceding result is very general in the sense that it applies to a wide class of solutions to Landau-Lifschitz equations. To go further and obtain stronger results (like stabilization and convergence to minimizers), we need stronger solutions. This is precisely the aim of the following Sections.

9.4 Global PDE smooth solutions

In this section, we investigate the existence and uniqueness of global smooth solutions of the Landau-Lifschitz equation (9.8) on the domain $\Omega_\lambda := \sqrt{\lambda}\Omega$, with $\lambda > 0$. As already explained, it is equivalent to study (9.18,9.19). First, we show existence and uniqueness of local (in time) smooth solutions.

Theorem 37 *Let Ω be a bounded regular open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, $\alpha > 0$, $\lambda > 0$, $H_{ext} \in C^0(\mathbb{R}_+, \mathbb{R}^3)$ and $m_0 \in H^2(\Omega, S^2)$ be such that $\frac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$. There exist a time $T^* = T^*(\Omega, \alpha, \lambda, \|m_0\|_{H^2(\Omega)}, \|H_{ext}\|_{L^\infty})$ and a unique*

$$m \in C^0([0,T], H^2(\Omega, S^2)) \cap H^1((0,T), H^1(\Omega, S^2)) \cap L^2((0,T), H^3(\Omega, S^2)),$$

for all $T \in (0, T^*)$, satisfying (9.18) and (9.19). Moreover, such regular solutions depend continuously on m_0 for the topology $C^0([0,T], H^2(\Omega, S^2))$.

In the 2D case, this result can be improved since global existence holds for small λ (*i.e.* small domains $\Omega_\lambda = \sqrt{\lambda}\Omega$), with initial conditions m_0 in a H^1-neighborhood of constants, and for all (bounded and regular) domains Ω.

Theorem 38 *Let Ω be a bounded open subset of $\mathbb{R}^2$, $\alpha > 0$, $\lambda > 0$, $H_{ext} \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$ with $\dot{H}_{ext} \in L^1(\mathbb{R}_+, \mathbb{R}^3)$. There exists $C^*(\Omega) > 0$ such that for every $m_0 \in H^2(\Omega, S^2)$ with $\frac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$ and*

$$\int_\Omega |\nabla m_0|^2 + \lambda \left(\int_\Omega |H_d(m_0)|^2 + 4\|H_{ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)} + 2\|\dot{H}_{ext}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)} \right) < \frac{1}{C^*(\Omega)}, \qquad (9.22)$$

the smooth solution of (9.18) exists on $\mathbb{R}_+$.

In the 3D case, the result of Theorem 37 can also be improved, when Ω is an ellipsoidal domain. We get in that case global existence of smooth solutions for small λ (*i.e.* small ellipsoids $\Omega_\lambda = \sqrt{\lambda}\Omega$), and for initial conditions m_0 in a H^2-neighborhood of constants.

Theorem 39 *Let Ω be a 3D ellipsoid domain. There exists $C^{**}(\Omega) > 0$ such that, for every $\alpha > 0$, for every $\lambda \in (0,1)$, for every $H_{ext} \in C^0 \cap L^\infty(\mathbb{R}_+, \mathbb{R}^3)$, for every $m_0 \in H^2(\Omega, S^2)$ with $\frac{\partial m_0}{\partial \nu} = 0$ on $\partial\Omega$ that satisfy*

$$C^{**}(\alpha + 1)(1 + \|H_{ext}\|_{L^\infty(\mathbb{R}_+)}) \leqslant \frac{\alpha}{\lambda}, \qquad (9.23)$$

$$C^{**}(\alpha+1)[\|\Delta m_0\|_{L^2}+\|\Delta m_0\|_{L^2}^2]<\alpha, \tag{9.24}$$

the smooth solution of (9.18) exists on $\mathbb{R}_+$ *and verifies, for every* $T>0$

$$\|\Delta m(T)\|_{L^2}^2+\frac{\alpha-N(T)}{\lambda}\int_0^T\|\nabla\Delta m(t)\|_{L^2}^2dt\leqslant\|\Delta m_0\|_{L^2}^2, \tag{9.25}$$

where

$$N(T):=\sup\left\{C^{**}(\alpha+1)[\|\Delta m(t)\|_{L^2}+\|\Delta m(t)\|_{L^2}^2];t\in[0,T]\right\}. \tag{9.26}$$

In particular, we have

$$\|\Delta m(t)\|_{L^2}\leqslant\|\Delta m_0\|_{L^2}\ ,\ \forall t>0. \tag{9.27}$$

Remark 16 *As a corollary of Theorem 38, we have the existence of 2D global smooth solutions for (9.18) when* λ *is small enough, namely*

$$\lambda\left(1+4\|H_{ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)}+2\|\dot{H}_{ext}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)}\right)<\frac{1}{2C^*(\Omega)},$$

and for every initial data $m_0\in H^2(\Omega,\mathbb{R}^3)$ *which satisfies* $\frac{\partial m_0}{\partial\nu}=0$ *on* $\partial\Omega$ *and*

$$\int_\Omega|\nabla m_0|^2<\frac{1}{2C^*(\Omega)}.$$

This defines an H^1*-neighborhood of uniform magnetizations of fixed size. Similarly, a Corollary of Theorem 39 is the existence of 3D global smooth solutions of (9.18) when* λ *is small enough, namely*

$$\lambda\leqslant\frac{\alpha}{C^{**}(\alpha+1)(1+\|H_{ext}\|_{L^\infty(\mathbb{R}_+)})},$$

and for every $m_0\in H^2(\Omega,\mathbb{R}^3)$ *which satisfies* $\frac{\partial m_0}{\partial\nu}=0$ *on* $\partial\Omega$ *and*

$$\|\Delta m_0\|_{L^2}<\min\{1,\frac{\alpha}{2C^{**}(\alpha+1)}\}.$$

This defines an H^2*-neighborhood of uniform magnetizations of fixed size.*

Remark 17 *Theorem 38 provides the existence of global smooth solutions of (9.18) when* λ *is small enough and for initial conditions in a* H^1*-neighborhood of any minimizer* m_λ^* *of the micromagnetic energy* $\mathcal{E}_\lambda$*, defined by (9.20), when* $\lambda>0$ *is small enough. Indeed,*

$$\|\nabla m_\lambda^*\|_{L^2(\Omega)}^2\leqslant2\lambda\mathcal{E}_\lambda(m_\lambda^*)\leqslant2\lambda\mathcal{E}_\lambda(e_1\chi_\Omega)=2\lambda\|H_d(e_1\chi_\Omega)\|_{L^2(\Omega)}^2\leqslant2\lambda|\Omega|,$$

thus (9.22) holds when

$$\|\nabla(m_0-m_\lambda^*)\|_{L^2(\Omega)}^2+2\lambda[1+2\|H_{ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)}+\|\dot{H_{ext}}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)}]<\frac{1}{C^*(\Omega)}.$$

Therefore, Theorem 38 provides the existence of global smooth solutions of (9.18) when λ *is small enough, namely*

$$2\lambda[1+2\|H_{ext}\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^3)}+\|\dot{H}_{ext}\|_{L^1(\mathbb{R}_+,\mathbb{R}^3)}]<\frac{1}{2C^*(\Omega)},$$

and for initial conditions in a H^1-neighborhood of m_λ^*, namely

$$\|\nabla(m_0 - m_\lambda^*)\|_{L^2(\Omega)}^2 \leqslant \frac{1}{2C^*(\Omega)}.$$

Theorem 38 also provides the existence of global smooth solutions of (9.18) for initial conditions in an H^2-neighborhood of any minimizer of the micromagnetic energy $\mathcal{E}_\lambda$, defined by (9.20), when $\lambda > 0$ is small enough. Indeed, on a 3D ellipsoidal domain, for λ small enough, the minimizers of the micromagnetic energy $\mathcal{E}_\lambda$ are constant in space (see Proposition 11 below).

Remark 18 *In 3D, the existence of global solutions is only proven on ellipsoidal domains on which the stray field has a particular structure. Indeed, on such domains, one has*

$$H_d(m) = H_d(m_\sharp \chi_\Omega) + H_d(m - m_\sharp \chi_\Omega)$$

where $H_d(m_\sharp \chi_\Omega)$ is constant over Ω. In particular, we have the following inequality that will be crucial in the proof

$$\|\nabla H_d(m)\|_{L^2} \leqslant C\|\nabla m\|_{L^2}.$$

These results are more general than [50, Theorems 1.1-4] :

- in Theorem 37, we take into account an external field H_{ext} which is not considered in [50, Theorems 1.1, 1.2],
- in Theorem 38, we take into account the stray field $H_d(m)$ and the external field H_{ext} which are not considered in [50, Theorems 1.3, 1.4],
- Theorem 39 deals with global solutions in a 3D case, this situation is not investigated in [50].

The proofs of Theorems 37 and 38 follow the ones of [50]. The proof of Theorem 39, instead, involves different arguments.

9.5 Exponential stabilization of uniform magnetizations on ellipsoidal domains

The goal of this subsection is to propose external magnetic fields H_{ext} that produce exponential convergence to global minimizers of the energy $\mathcal{E}_\lambda$. We consider an ellipsoidal domain Ω of $\mathbb{R}^3$ with $|\Omega| = 1$, $\alpha > 0$ and we study (9.18), (9.19) with $\lambda > 0$. On Ω, the stray field generated by a uniform magnetization is constant, thus, up to a change of coordinates, we may assume that

$$\forall x \in \Omega,\ \forall \widetilde{m} \in S^2,\ H_d(\widetilde{m}\chi_\Omega)(x) = -D\widetilde{m}, \tag{9.28}$$

where

$$D = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},\ 0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3. \tag{9.29}$$

Therefore, for non uniform magnetizations the stray field is given by

$$H_d(m) = -Dm_\sharp + \widetilde{H}_d(m), \tag{9.30}$$

where

$$\widetilde{H}_d(m) := H_d(m - m_\sharp \chi_\Omega). \tag{9.31}$$

We are now in a position to state the results. We begin with the description of the minimizers.

Proposition 11 *Let Ω be a 3D ellipsoid. There exists $\lambda^* = \lambda^*(\Omega) > 0$ such that, for every $\lambda \in (0, \lambda^*)$, the micromagnetic energy $\mathcal{E}_\lambda$ has exactly two global minimizers : $m \equiv e_1$ and $m \equiv -e_1$.*

Physically speaking, it is clear that taking H_{ext} parallel to e_1 should force the magnetization to converge to e_1. This is indeed the case, and we even show a slightly stronger result. More precisely, we prove that, for λ small enough (*i.e.* for small domains $\Omega_\lambda = \sqrt{\lambda}\Omega$), the constant external field $H_{ext} = \beta e_j$ forces, locally around e_j, the exponential convergence of the PDE solutions to e_j, when the parameter $\beta > 0$ is large enough. When $j = 1$, we therefore get the exponential stabilization of the global minimizers of the energy.

Theorem 40 *Let Ω be a 3D ellipsoid and $\alpha > 0$. Let $\alpha_1, \alpha_2, \alpha_3, \beta_1^*, \beta_2^*, \beta_3^*$ be the real numbers defined by (9.28,9.29) and*

$$\beta_1^* := \alpha_1 + \tfrac{\alpha_3 - \alpha_2}{2\alpha}, \quad \beta_2^* := \alpha_2 + \tfrac{\alpha_3 - \alpha_1}{2\alpha}, \quad \beta_3^* := \alpha_3 + \tfrac{\alpha_2 - \alpha_1}{2\alpha}. \tag{9.32}$$

Let $j \in \{1, 2, 3\}$ and $\beta > \beta_j^*$. There exists $\lambda^* = \lambda^*(\Omega, \alpha, \beta) > 0$, $\eta = \eta(\Omega, \alpha) > 0$, $\nu = \nu(\Omega, \alpha, \beta, \lambda) > 0$, $K(\Omega, \alpha, \beta, \lambda) > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ with $\frac{\partial m_0}{\partial \nu} \equiv 0$ on $\partial\Omega$, $E_{\beta,j}(m_0) \leqslant \beta$ and

$$\|\Delta m_0\|_{L^2} < \eta \tag{9.33}$$

there exists a unique global solution of (9.18), (9.19) with $H_{ext} \equiv \beta e_j$ which satisfies

$$\|m(t) - e_j\|_{H^1(\Omega)} \leqslant K \|m_0 - e_j\|_{H^1(\Omega)} e^{-\nu t}. \tag{9.34}$$

In **(P1)** [11], we also prove that, $H_{ext} = 0$ ensures locally the convergence to e_1, but the rate of convergence is not known.

Theorem 41 *Let Ω be a 3D ellipsoid and $\alpha > 0$. There exist $\lambda^* = \lambda^*(\Omega, \alpha) > 0$, such that, for every $\lambda \in (0, \lambda^*)$, there exists $\eta = \eta(\Omega, \alpha, \lambda) > 0$, such that, for every $m_0 \in H^2(\Omega, S^2)$ with*

$$\frac{\partial m_0}{\partial \nu} \equiv 0 \text{ on } \partial\Omega \tag{9.35}$$

and

$$\|m_0 - e_1\|_{H^2} < \eta, \tag{9.36}$$

there exists a unique global smooth solution of (9.18)-(9.19) with $H_{ext} \equiv 0$ and it satisfies

$$\|m(t) - e_1\|_{H^s} \to 0 \text{ when } t \to +\infty, \forall s < 2.$$

9.6 Magnetization switching on ellipsoidal domains : PDE study.

We use the notation β_1^* defined in (9.32), $H_{ext}(m)$ defined by (9.13).

Proposition 12 *Let Ω be a 2D (resp. 3D) ellipsoid domain, $\alpha > 0$, $\beta > \beta_1^*$, $\lambda_* = \lambda_*(\Omega, \alpha, \beta)$ be as in Theorem 40, $T > 0$, $m_{ref} \in H^2((0,T), S^2)$ be such that $-m_{ref}(0) = m_{ref}(T) = -e_1$. We define $H_{ext} \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$ by*

$$H_{ext}(t) := \begin{cases} H_{ext}(m_{ref}(t)) & \text{if } 0 \leqslant t \leqslant T, \\ \beta e_1 & \text{if } t > T, \end{cases}$$

where $H_{ext}(m_{ref})$ is defined by (9.13). There exists $\eta > 0$ such that, for every $m_0 \in H^2(\Omega, S^2)$ (resp. $m_0 \in H^3(\Omega, S^2)$ with $\frac{\partial m_0}{\partial \nu} \equiv 0$ on $\partial\Omega$ and $\|m_0 + e_1\|_{H^1(\Omega)} < \eta$, (resp. $\|m_0 + e_1\|_{H^2(\Omega)} < \eta$) the solution of (9.18) converges exponentially to e_1 in $H^1(\Omega)$.

The proof is extremely simple : it relies on the continuity with respect to initial conditions for the $C^0([0, T], H^2)$-topology and Theorem 40 on $(T, +\infty)$.

9.7 Conclusion, open problems, perspectives

In this chapter, we have tackled the problem of magnetization switching with either a tridimensional external magnetic field or a bidimensional one, or a spin polarized current. In both cases, we have studied the controllability of the ODE subsystem, giving a complete answer to the problem. We have then extended the results to the full PDE model (Landau-Lifschitz-Gilbert equations) in the case of small enough ellipsoidal domains. Although only very few results are available for the full PDE in all generality, we have been able to use the fact that in small enough particles the magnetization remains almost constant in space for all time. This is however not the case for the sizes of particles used in current devices [155], and a more complicated space structure occurs, although minimizers are still almost constant in space.

In this chapter, we only used the natural dissipation of the system. Another research direction is the rapid feedback stabilization of the minimizers of the energy : given a prescribed decay rate $\sigma > 0$, is it possible to find feedback laws ensuring, at least locally, an exponential convergence to the minimizer u, with an exponential rate σ.

Another question concerns the uniqueness (up to the transformation $u \mapsto -u$) of the global minimizer, in general.

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