

Large Time Asymptotics for Partially Dissipative Hyperbolic Systems

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Abstract

This work is concerned with (n -component) hyperbolic systems of balance laws in m space dimensions. First, we consider linear systems with constant coefficients and analyze the possible behavior of solutions as $t \rightarrow \infty$. Using the Fourier transform, we examine the role that control theoretical tools, such as the classical Kalman rank condition, play. We build Lyapunov functionals allowing us to establish explicit decay rates depending on the frequency variable. In this way we extend the previous analysis by Shizuta and Kawashima under the so-called algebraic condition (SK). In particular, we show the existence of systems exhibiting more complex behavior than the one that the (SK) condition allows. We also discuss links between this analysis and previous literature in the context of damped wave equations, hypoellipticity and hypocoercivity. To conclude, we analyze the existence of global solutions around constant equilibria for nonlinear systems of balance laws. Our analysis of the linear case allows proving existence results in situations that the previously existing theory does not cover.

1. Introduction and main results

1.1. Problem formulation

This work is concerned with the following n -component hyperbolic system of balance laws in m space dimensions:

$$\frac{\partial w}{\partial t} + \sum_{j=1}^m \frac{\partial F_j(w)}{\partial x_j} = Q(w). \quad (1)$$

Here $m, n \in \mathbb{N}^*$, $w : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $w = w(t, x)$ is the unknown and $Q, F_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are smooth functions.

Such nonlinear systems typically govern nonequilibrium processes in physics for media with hyperbolic response as, for example, in gas dynamics. They also arise in the numerical simulation of conservation laws by relaxation schemes (see [1,3,12,37] and references cited therein). In many applications, the source term $Q(w)$ has, or can be transformed by a linear transformation into, the form

$$Q(w) = \begin{pmatrix} 0 \\ q(w) \end{pmatrix} \tag{2}$$

with $0 \in \mathbb{R}^{n_1}$, $q(w) \in \mathbb{R}^{n_2}$, where $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$.

It is well known that (1) has local (in time) smooth solutions (see [23,29]), but these solutions may develop singularities (that is shock waves) in finite time, even when the initial data are smooth and small (see [10,29]). However, in many physical examples, thanks to the interplay between the source term and the flux, there exist global smooth solutions for a suitable set of initial conditions. Total dissipation, which consists of requiring a source damping term to enter into each of the equations of the system distributed over the entire space, is a well known assumption for global existence for suitable classes of initial data (see [22]). But this condition is too strong, and it is not satisfied by systems (1)–(2) with $n_1 \neq 0$ (in which the dissipation is not present in all the components of the system) and more general systems with relaxation (see [9,25]). This is the case for the isentropic Euler system with damping

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial f(u)}{\partial x} = -v, \tag{3}$$

with $f'(u) < 0$, which has been considered in [18,26]. In this case, the damping term, even if it enters only in the second equation, may prevent shock formation. We also refer to [27] where similar issues are addressed for systems with a linear principal part, which correspond, in particular, to damped wave equations involving nonlinear convective terms.

The existence of global smooth solutions for system (1)–(2) is one of the two main topics of this article. In fact, the study of the nonlinear systems (1) we shall develop relies on a linearization principle, around constant equilibria, and this requires analyzing partially dissipative linear hyperbolic systems of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^m A_j \frac{\partial w}{\partial x_j} = -Bw \tag{4}$$

where A_1, \dots, A_m, B are $n \times n$ real matrices, $A_j := (a_{k,l}^{(j)})_{1 \leq k,l \leq n}$ being symmetric, for $j = 1, \dots, m$, and B such that

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad D \in \mathbb{R}^{n_2 \times n_2}, \quad X^t D X > 0, \quad \forall X \in \mathbb{R}^{n_2} - \{0\}. \tag{5}$$

Note that D is not assumed to be symmetric. The form (4)–(5) was first introduced in [2] and it is, after a change of variables, the general form for the linearized operator of symmetrizable, strictly dissipative systems, in the sense of [13].

The analysis of linear systems of the form (4) is relevant, as mentioned above, to understanding the behavior of nonlinear systems and, as we shall see, they may exhibit a very rich behavior. Analysis of system (4) is the first goal of this article.

The solutions of (4) with initial conditions $w_0 \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ are explicit. Indeed, applying the Fourier transform in the x variable, system (4) can be rewritten as

$$\frac{\partial \hat{w}}{\partial t} + i \sum_{j=1}^m A_j \xi_j \hat{w} = -B \hat{w}, \tag{6}$$

or

$$\frac{\partial \hat{w}}{\partial t}(t, \xi) = E(\xi) \hat{w}(t, \xi) \tag{7}$$

where

$$E(\xi) := -B - iA(\xi), \quad A(\xi) := \sum_{j=1}^m \xi_j A_j. \tag{8}$$

Solving this first order ordinary differential equation, we get

$$\hat{w}(t, \xi) = \exp[E(\xi)t] \hat{w}^0(\xi). \tag{9}$$

Note that, when $n_2 \neq n$, the matrix $-E(\xi)$ is not coercive. Indeed, with the notation

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathbb{C}^n, \quad X_1 \in \mathbb{C}^{n_1}, \quad X_2 \in \mathbb{C}^{n_2}, \quad n_1 + n_2 = n,$$

we have

$$\Re[X^t E(\xi) \bar{X}] = -\Re[X_2^t D \bar{X}_2].$$

Thus, this quadratic form does not provide any information on the X_1 component.

However, it is by now well known in the context of linear finite-dimensional systems that this fact is not present an obstacle for the solutions of (6) to decay as $t \rightarrow \infty$. Indeed, the interaction of the dissipative operator B with the time-dynamics generated by (6) may eventually dissipate all components of the solutions. This can be viewed, for instance, through the Kalman rank condition for the control of finite-dimensional systems (see [7,38]), which we shall discuss in detail in Section 2.

This fact is also well known in several other contexts and, in particular, for dissipative wave equations [14, 15] which are particular instances of (4), and partially diffusive partial differential equations where the notions of hypoellipticity [16] and hypocoercivity [34] have been introduced to measure the global effect of partial diffusion in the regularity and the time decay of solutions, respectively.

As we shall see, under rather general assumptions on the matrices A_1, \dots, A_m and B and, more precisely, under the Kalman rank condition for the pair $(A(\xi), B)$, it can be proved that

$$\exists C > 0, \lambda(\xi) > 0 : \exp[E(\xi)t] \leq C e^{-\lambda(\xi)t}. \tag{10}$$

This decay property, together with explicit estimates on the dependence of $\lambda(\xi)$ with respect to ξ , allows us to accurately describe the asymptotic behavior of solutions of (4) as $t \rightarrow \infty$, and to derive a decomposition of solutions in which various terms decaying with different rates can be distinguished. Obviously, the overall picture depends in a critical way on the properties of the function $\xi \rightarrow \lambda(\xi)$. The analysis of this function is a complex issue to which we will devote a significant part of this article.

There is in fact an extensive literature on the subject. For instance, in [30], the authors study systems (4) under the so-called Shizuta–Kawashima (SK) condition:

$$(SK) \quad \forall \xi \in \mathbb{R}^m - \{0\}, \text{Ker}(B) \cap \{\text{eigenvectors of } A(\xi)\} = \{0\}, \quad (11)$$

and prove that

$$\exists C, c > 0 \text{ s. t. } \forall \xi \in \mathbb{R}^m, \exp[E(\xi)t] \leq C e^{-c \min\{1, |\xi|^2\}t}. \quad (12)$$

Thanks to (9) and (12), the authors deduce that any solution w of (4) associated with an initial condition $w_0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n)$ can be decomposed as

$$w = w_1 + w_2 \quad (13)$$

where

$$\begin{aligned} \|w_1(t)\|_{L^2(\mathbb{R}^m, \mathbb{R}^n)} &\leq C e^{-\lambda t} \|w^0\|_{L^2(\mathbb{R}^m, \mathbb{R}^n)}, \quad \forall t \in (0, +\infty), \\ \|w_2(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} &\leq C t^{-\frac{m}{2}} \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}, \quad \forall t \in (0, +\infty), \end{aligned} \quad (14)$$

and C, λ are positive constants depending only on A_1, \dots, A_m, B . The two components w_1 and w_2 correspond, respectively, to the high and low frequency components. The high frequency component decays exponentially while the low frequency component decays polynomially with the decay rate of the heat kernel. We also refer to [6] and [27] for similar results for damped wave equations.

This result has motivated many other inquiries. For instance, in [2], the authors proved more precise decay rates for the high frequency component w_1 under (SK), and extended the analysis for nonlinear systems.

Estimate (12) is equivalent to saying that $\lambda(\xi) \geq c \min\{1, |\xi|^2\}$ in (10). In particular, (SK) implies that $\lambda(\xi)$ may only degenerate quadratically at $\xi = 0$. Roughly speaking, (SK) is a natural sufficient condition to guarantee that the damping term affects all the components of the system and the L^2 -decay of the solutions of (4) as $t \rightarrow +\infty$. But it is not sharp. Indeed, as we shall see, there are many situations in which $\lambda(\xi)$ degenerates on points other than $\xi = 0$; but still the L^2 -decay of solutions holds, together with decompositions in the spirit of (13), but involving extra terms, decaying more slowly than the m -dimensional heat kernel. To elucidate this, we develop a careful analysis of the finite-dimensional behavior of system (6) in terms of the multi-dimensional parameter ξ . This analysis is inspired by control theoretical tools. Indeed, our approach starts from the observation that (SK) holds if the pair of matrices $(A(\xi), B)$ satisfies the Kalman rank condition:

$$\text{rk}[B, A(\xi)B, \dots, A(\xi)^{n-1}B] = n.$$

A similar observation is made in the context of hypocoercivity by Villani [35, Remark 17, p. 25], who attributes the remark to D. Serre.

The key ingredient in obtaining a complete decomposition of solutions is to obtain a careful measure of the decay rate $\lambda(\xi)$ of solutions of (6) as a function of ξ . To do this we construct explicit Lyapunov functionals, taking advantage of the interaction of the matrix B with the dynamics generated by $A(\xi)$. This kind of Lyapunov function is similar to those introduced by Villani [34] for the analysis of the decay of partially diffusive systems, and is also linked, as we mentioned above, to the extensive literature on damped wave equations (see [15, 27], for instance).

The approach we develop here, in addition to being more systematic, also has the added advantage of being simpler to carry out technically, since we avoid some of the long developments in [30] for proving the sufficiency of (SK) in achieving the decomposition above.

Let us now return to the nonlinear systems of balance laws (1). The existence of global smooth solutions in the neighborhood of a constant equilibrium, $W_e \in \mathbb{R}^n$ and $Q(W_e) = 0$, was proved in [36], under a suitable entropy dissipation assumption, when the linearized system around W_e satisfies (SK). We also mention [13] for the same result in one space dimension ($m = 1$) with examples of applications.

The techniques we develop in this article, in the frame of [36], allow us to obtain explicit estimates of the size of the neighborhood of W_e where the existence of global smooth solutions holds. As a consequence of this, using an argument inspired in Coron's return method [7], we are able to prove a global existence result around a constant equilibrium that does not fulfill the (SK) condition. We are also able to prove uniform well posedness results with respect to a relaxation parameter.

Before the papers [36] and [13], which are of general breadth, there were a few papers restricted to specific examples, such as [18, 19, 31]. In addition, some papers deal with the global existence and/or the asymptotic stability of weak entropy solutions, still under (SK) (see [11, 28, 33]).

Finally, let us give some references regarding the problem that go beyond the (SK) condition.

In [17, 20], there are interesting ideas about estimates when the (SK) condition fails, which are obtained with a loss of regularity with respect to the initial data.

One of the main examples of a physical system violating (SK) is the Kerr–Debye model of nonlinear Maxwell equations. Concerning the global existence for this nonlinear system, some recent results have been obtained in one dimension in [5], where the (SK) condition is violated when perturbing a zero electric field, and in three dimensions in [4], where the (SK) condition is violated almost all the time (see also references therein).

Finally, in [24] a global existence result has been established for a whole class of systems without assuming the (SK) condition.

1.2. Main results

Let us now describe the contents of this article in more detail. Sections 2, 3, 4 and 5 are devoted to analysis of partially dissipative linear hyperbolic systems, and Section 6 is devoted to nonlinear systems of balance laws.

More precisely, Section 2 is concerned with the Kalman rank condition, the (SK) condition, the notion of hypocoercivity and decay rates. Section 2.1 is devoted to a preliminary discussion in which we show that (SK) is equivalent to the classical Kalman rank condition in control theory for the pairs $(A(\xi), B)$ and all $\xi \neq 0$. Then we develop the proof of decay (10) using Lyapunov functionals (see Section 2.2). In particular, this yields a more systematic approach and a simpler way to get the decomposition (13)–(14) in [30] under the condition (SK) (see Section 2.3).

In Section 3, using the tools developed in Section 2, we investigate the asymptotic behavior for (4) without the (SK) condition. More precisely, we study the L^2 -stability of (4). To do that we introduce the *set of degeneracy*

$$\mathcal{D}(A_1, \dots, A_m, B) := \{\xi \in \mathbb{R}^m : \text{Ker}(B) \cap \{\text{eigenvectors of } A(\xi)\} \neq \{0\}\}, \quad (15)$$

That is, the set of values of ξ for which (SK) fails.

Under the (SK) condition this set is the trivial one, $\{0\}$. But, of course, this is false in general and, as we shall see, the structure of this set determines the asymptotic behavior of solutions. In Section 3.2, using the fact that the Kalman condition fails on the set of degeneracy when (SK) does not hold, we prove that, when (SK) does not hold, then, either

- $\mathcal{D}(A_1, \dots, A_m, B)$ is a strict algebraic submanifold of \mathbb{R}^m ,
- or
- $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$.

In Section 3.3, we prove that, in the first case, (4) is strongly stable in L^2 , that is, all L^2 solutions tend to zero in L^2 as $t \rightarrow \infty$. In Section 3.4, we prove that, in the second case, there exist nondissipated solutions of constant L^2 -norm.

In Section 3.5, we study the one-dimensional case $m = 1$, whatever the size n of the system may be. We show that, in this case, the (SK) condition is sharp in the sense that it completely characterizes the behavior of the solutions:

- (SK) is a necessary and sufficient condition for the strong stability in $L^2(\mathbb{R}, \mathbb{R}^n)$,
- (SK) is a necessary and sufficient condition for the decomposition (14),
- without (SK), any solution associated to an L^2 initial condition can be decomposed as $w = w_1 + w_2 + w_{trw}$ where w_1, w_2 satisfy (14) and w_{trw} is the sum of a finite number of traveling waves.

This shows, in particular, that in one dimension we may not have a third component decaying at infinity with a slower decay rate. The latter can arise only in the multi-dimensional case.

In Section 4, we investigate the analogue of the decomposition (13)–(14) for the solutions of (4), when (SK) is not fulfilled and when the measure of the set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes. We answer this question in some particular cases where the set of degeneracy is the union of a finite number of vector subspaces.

In Section 4.1, we prove a general statement giving a decomposition for the solutions of (4) when the set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ is the union of a

finite number of vector subspaces and under an assumption of the following type

$$N_*(\omega) \geq C \text{dist}(\omega, \mathcal{D}(A_1, \dots, A_m, B))^\alpha, \quad \forall \omega \in S^{m-1}, \quad (16)$$

for some constants $C > 0$ and $\alpha \geq 2$, where

$$N_*(\omega) := \min \left\{ \sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1} \right\}, \quad (17)$$

for a suitable small enough $\epsilon > 0$ and suitable exponents m_k (see Proposition 1). Note that the value of $N_*(\omega)$ provides a quantitative version of the (SK) property or the Kalman rank property, in the sense that, when it fails on an isolated point ω , it holds in the neighboring points (that do not belong to $\mathcal{D}(A_1, \dots, A_m, B)$), with an explicit lower bound on N_* as a power of the distance function.

This decomposition is of the form $w = w_1 + w_2 + w_3 + w_4$, where w_1 and w_2 satisfy (14) and w_3 (resp. w_4) contains some (but not all) high (resp. low) frequencies and decays like $t^{-1/\alpha}$ (resp. $t^{-r/\alpha}$ where $r \in \mathbb{N}^*$).

In this way we see that, in dimension $m \geq 2$, there is a whole class of phenomena that do not arise under the (SK) condition. Note that we already obtained a complete classification of the possible decompositions for the solutions of (4) when $m = 1$ in Section 3.5.

In Section 4.2, we study the particular case $n_1 = 1$ (that is $\text{Ker}(B) = \text{Span}(e_1)$) in which the condition on the set of degeneracy being the union of vector subspaces is automatically satisfied. In that case, (16) holds with a smaller exponent $\alpha = 2$. This leads to a decomposition for the solutions of (4), in this particular case $n_1 = 1$.

In Section 4.3, considering explicit examples, we show that there are situations in which $\alpha = 4$ is the smallest exponent one can have in (16), and the parameter r may take any positive value.

In Section 4.4, we deduce from the previous analysis a complete classification in the case $n = 2$, whatever the space dimension m is.

In Section 5, we recapitulate the various results obtained in the previous sections, presenting them in a table. This classification is still incomplete in the sense that not all possible values of m and n are covered by our analysis. Indeed, we do not provide a decomposition

- in the general case where the set of degeneracy is the union of a finite number of vector subspaces because we only study particular examples,
- when the set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ is an algebraic submanifold that is not a union of vector subspaces, or
- when $\mathcal{D}(A_1, \dots, A_m, B)$ is the whole space.

We also make precise some open questions and some conjectures about the cases which our partial classification does not cover.

In Section 6, we study the existence of global (in time) smooth solutions for the nonlinear system (1), locally around a constant equilibrium $W_e \in \mathbb{R}^n$. In Section 6.1, we describe more precisely the context of our work. The novelty of our study with respect to [36] is that we do not impose the (SK) condition on the linearized system around W_e . To do that we proceed in two steps. In Section 6.2, we first

assume that (SK) holds and, taking advantage of the explicit Lyapunov function introduced in Section 2.2, we make the result in [36] more precise by giving an explicit estimate on the size for the neighborhood of W_e on which global existence holds. Then, in Section 6.3, using these explicit estimates and under some suitable assumptions on the nonlinearities, we prove the existence of global solutions for (1), around a constant equilibrium W_e that does not satisfy (SK). This is done assuming the existence of a constant equilibrium fulfilling (SK) close enough to W_e . This approach is inspired by Coron's return method for the controllability of nonlinear systems (see [7] for an introduction and examples of applications), that takes advantage of the nonlinearity of the system. We conclude Section 6.3 with an example of an application of the foregoing theorem.

In Section 6.4, we give a second application of Section 6.2. Precisely, we consider a class of nonlinear hyperbolic system with relaxation, and we prove its uniform well-posedness with respect to the relaxation parameter (which is useful when one wants to study the limit of the solutions when the relaxation parameter tends to zero). This result may be viewed as a generalization of the equivalent one in [8].

At this point, it is convenient to note that the fact that (SK) was not necessary for the existence of global smooth solutions around a constant equilibrium was already known. Indeed, Zeng proved in [37] the existence of global one-dimensional ($m = 1$) smooth solutions for an equation of gas dynamics around a constant state, without the (SK) condition, splitting the system in two parts, one of them being linearly degenerate. More recent works [4,5] focus on global existence for the nonlinear Kerr-Debye model, which does not satisfy (SK). Finally, in [24] a global existence result has been established for a whole class of systems without assuming the condition (SK).

Our goal in Section 6 is to show that, thanks to our explicit Lyapunov functional for the linearized system, one may improve the known results for nonlinear systems.

Notation. In this article, $\Re(\cdot)$, $\Im(\cdot)$ denote the real and imaginary parts of complex numbers, (e_1, \dots, e_n) is the canonical basis of \mathbb{R}^n and $\langle \cdot, \cdot \rangle$ denotes the hermitian product in \mathbb{C}^n , $\langle X, Y \rangle = X^t \bar{Y}$.

The notation given in this introduction ($m, n, n_1, n_2, A_j = (a_{k,l}^{(j)})_{1 \leq k,l \leq n}$, $B, D, \mathcal{D}(A_1, \dots, A_m, B)$) will be used throughout the paper.

2. Rank conditions, (SK), hypocoercivity and decay rates

In Section 2.1, we show the equivalence between the (SK) condition for hyperbolic systems and the Kalman rank condition in control theory. In Section 2.2, taking advantage of this equivalence, we prove an explicit decay rate for $\lambda(\xi)$ in (10), by means of a Lyapunov functional. Finally, in Section 2.3, we recover Shizuta-Kawashima's decomposition (13)–(14) for the solutions of (4) under the (SK) condition.

2.1. (SK) and Kalman rank conditions

The following lemma is one of the key ingredients of this section.

Lemma 1. *Let $n \in \mathbb{N}^*$, A, B be $n * n$ matrices with real coefficients, such that B has the form (5). The following statements are equivalent*

- (1) *A and B satisfy (SK) : $\{\text{eigenvectors of } A\} \cap \text{Ker}(B) = \{0\}$,*
- (2) *$\forall y \in \mathbb{C}^n - \{0\}$, $t \mapsto B \exp(At)y$ does not vanish on \mathbb{R} ,*
- (3) *$\forall y \in \mathbb{C}^n - \{0\}$, there exists $k \in \{0, 1, \dots, n - 1\}$ such that $BA^k y \neq 0$,*
- (4) *for every $a_0, \dots, a_{n-1} > 0$, the expression*

$$N(y) := \left(\sum_{k=0}^{n-1} a_k |BA^k y|^2 \right)^{1/2}$$

defines a norm on \mathbb{C}^n

- (5) *(A, B) satisfies the Kalman rank condition: the $(n^2) * n$ Kalman matrix*

$$K := \begin{pmatrix} B \\ BA \\ \dots \\ BA^{n-1} \end{pmatrix} \tag{18}$$

has rank n .

These equivalences are classical (see for example [7, Chapter 1.3], [32, Theorem 2.2.1]). We give a proof for the sake of completeness.

Proof of Lemma 1. Let us prove “(1) \Rightarrow (2)”. We assume that (2) is false. Then, there exists $y \in \mathbb{C}^n - \{0\}$ such that

$$B \exp(At)y \equiv 0. \tag{19}$$

Let

$$\det(A - XI_n) := \prod_{j=1}^v (X - \lambda_j)^{r_j}$$

be the characteristic polynomial of A . We have

$$\begin{aligned} \mathbb{C}^n &= \bigoplus_{j=1}^v \text{Ker}[(A - \lambda_j)^{r_j}] \\ y &= \sum_{j=1}^v y_j. \end{aligned}$$

The equality (19) gives

$$\sum_{j=1}^v e^{\lambda_j t} \sum_{k=0}^{r_j-1} \frac{t^k}{k!} B(A - \lambda_j)^k y_j \equiv 0.$$

Thanks to the linear independence of the family

$$\{t^k e^{\lambda_j t}; 0 \leq k \leq r_j - 1, 1 \leq j \leq \nu\}$$

we deduce that

$$B(A - \lambda_j)^k y_j = 0, \quad \forall k \in \{0, \dots, r_j - 1\}, \quad \forall j \in \{1, \dots, \nu\}.$$

Let $j \in \{1, \dots, \nu\}$ be such that $y_j \neq 0$ and $k \in \{0, \dots, r_j - 1\}$ be the largest integer such that $(A - \lambda_j)^k y_j \neq 0$. Then, $(A - \lambda_j)^k y_j$ is an eigenvector of A in the null space of B . Thus, **(1)** is false.

Let us prove now that “**(2)**⇒**(3)**”. We assume that **(3)** is false. Then, there exists $y \in \mathbb{C}^n - \{0\}$ such that $BA^k y = 0$ for $k = 0, \dots, n - 1$ and thanks to the Cayley–Hamilton theorem, $BA^k y = 0$ for every $k \in \mathbb{N}$. Thus $B \exp(At)y \equiv 0$ and **(2)** is false.

The implication “**(3)**⇒**(4)**” is clear.

Let us prove “**(4)**⇒**(5)**”. We assume **(4)**. If $y \in \text{Ker}(K)$, then $BA^k y = 0$ for $k = 0, \dots, n - 1$. Thus $N(y) = 0$ and so $y = 0$. This proves that K is injective and thus $\text{rk}(K) = n$.

Finally, let us prove “**(5)**⇒**(1)**”. We assume **(1)** is false. Let $v \in S^{n-1}$ be an eigenvector of A in the null space of B . Then $Ky = 0$ and thus K is not injective. Therefore $\text{rk}(K) < n$, that is, **(5)** is false. \square

2.2. Lyapunov functionals and explicit decay rates

According to the results of the previous section, there is a very explicit connection between the (SK) condition, which arises naturally in the analysis of the decay of partially dissipative hyperbolic systems, and the Kalman rank condition in finite dimensional control theory.

In this subsection we develop one of the key elements of this article, which consists in deriving explicit decay rates for the finite-dimensional system (6) in terms of ξ . To do this we construct explicit Lyapunov functionals, which are modeled on those rank conditions and more precisely in the statement (4) of Lemma 1.

We set $\xi = \rho\omega$, with $\rho > 0$ and $\omega \in S^{m-1}$, and, consequently, rewrite (6) in the form:

$$\dot{x} = -(B + i\rho A(\omega))x, \quad x(0) = x_0 \in \mathbb{C}^n. \tag{20}$$

Proposition 1. *We fix a family of nonnegative real numbers $(m_k)_{0 \leq k \leq n}$ such that*

$$0 = m_0 < m_1 < \dots < m_n, \tag{21}$$

$$m_k - \frac{m_{k-1} + m_{k+1}}{2} \geq \delta > 0, \quad \forall k = 1, \dots, n - 1, \tag{22}$$

for some $\delta > 0$.

Let A_1, \dots, A_m, B be $n * n$ real matrices such that B has form (5). For $\omega \in S^{m-1}$, $\epsilon > 0$ we define the symmetric nonnegative matrix

$$M_\epsilon(\omega) := \sum_{k=0}^{n-1} \epsilon^{m_k} (A(\omega)^t)^k B^t B A(\omega)^k \tag{23}$$

and its minimal eigenvalue

$$N_{*,\epsilon}(\omega) := \min\{\langle x, M_\epsilon(\omega)x \rangle; x \in S^{n-1}\} = \min\left\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\right\}. \tag{24}$$

Then, there exist $\epsilon_* = \epsilon_*(A_1, \dots, A_m, B) \in (0, 1)$ and $\tilde{c} = \tilde{c}(A_1, \dots, A_m, B) > 0$ such that, for every $\epsilon \in (0, \epsilon_*)$, $x_0 \in \mathbb{C}^n$, $\rho \in (0, +\infty)$ and $\omega \in S^{m-1}$, the solution of (20) satisfies

$$|x(t)| \leq \sqrt{3}|x_0|e^{-\tilde{c}N_{*,\epsilon}(\omega)\min\{1,\rho^2\}t}, \quad \forall t \in (0, +\infty). \tag{25}$$

Remark 1. This result provides an exponential decay rate for $\exp[E(\rho\omega)t]$, which is explicit in terms of ρ and ω , and more precisely on $N_{*,\epsilon}(\omega)$. As we will see in Sections 2.3 and 4, the main interest of this Proposition is that it reduces the problem of the asymptotic behavior of the solutions of (4) to the study of the real valued map $\omega \in S^{m-1} \mapsto N_{*,\epsilon}(\omega) \in \mathbb{R}_+$.

Note that Proposition 1 holds without assuming the (SK) condition, which in fact enters only when trying to obtain a uniform lower bound on $N_{*,\epsilon}(\omega)$ for $\omega \in S^{m-1}$, in Section 2.3. In particular, it could be that for some value ω^* of ω , $N_{*,\epsilon}(\omega^*) = 0$. In that case, to get explicit decay rates for (4), one has to analyze the behavior of $N_{*,\epsilon}(\omega)$ for values ω close to ω^* . This fact will play an important role when deriving decompositions of solutions in the absence of the (SK) condition and analyzing their decay rates as $t \rightarrow \infty$, in Section 4.

Remark 2. The constant $\sqrt{3}$ in (25) can be replaced by any constant $\tilde{C} > 1$. Indeed, an easy adaptation of the proof below provides the following more general statement: for every $\tilde{C} > 1$, there exist $\epsilon_* = \epsilon_*(A_1, \dots, A_m, B) > 0$ and $\tilde{c} = \tilde{c}(\tilde{C}, A_1, \dots, A_m, B) > 0$ such that the same conclusion holds with $\sqrt{3}$ replaced by \tilde{C} .

On the other hand, \tilde{C} being fixed, it is natural to raise the issue of finding the value of ϵ that maximizes the decay rate. This is an open problem.

Remark 3. Notice that Proposition 1 holds without assuming the symmetry of the matrices A_1, \dots, A_m . This will be used in Section 6.

Proof of Proposition 1. First, let us introduce some notation. Since B has the form (5), there exists $C_1 = C_1(B) > 0$ such that

$$\Re\langle Bx, x \rangle \geq C_1|Bx|^2, \quad \forall x \in \mathbb{C}^n. \tag{26}$$

We consider the characteristic polynomial

$$\det(XI_n - A(\omega)) = X^n + \sum_{k=0}^{n-1} a_k(\omega)X^k,$$

$a_k(\omega)$ being the coefficient of its $k - th$ order term. We also set

$$\mathcal{M}_1 := \max\{\|BA(\omega)^k\| : 0 \leq k \leq n - 1, \omega \in S^{m-1}\}, \tag{27}$$

$$\mathcal{M}_2 := \max\{|a_k(\omega)| : 0 \leq k \leq n - 1, \omega \in S^{m-1}\}. \tag{28}$$

Let $\epsilon_* = \epsilon_*(A_1, \dots, A_m, B) > 0$ be small enough so that, for every $\omega \in S^{m-1}$,

$$\sum_{k=1}^{n-1} \epsilon_*^{m_k} \|(A(\omega)^t)^k B^t B A(\omega)^{k-1}\| < \frac{1}{2}, \tag{29}$$

$$\mathcal{M}_1 \epsilon_*^{m_1} + \epsilon_*^\delta + n \mathcal{M}_2^2 \epsilon_*^\delta < \frac{C_1}{2}, \tag{30}$$

$$\frac{\mathcal{M}_1^2(n-1)}{C_1} \epsilon_*^{m_1} + \epsilon_*^\delta (1 + n \mathcal{M}_2^2) < \frac{1}{2}. \tag{31}$$

For $\epsilon \in (0, \epsilon_*)$, $\rho \in (0, +\infty)$, $\omega \in S^{m-1}$ and $x \in \mathbb{C}^n$ we consider

$$\mathcal{L}_{\rho, \omega, \epsilon}(x) := |x|^2 + \min \left\{ \rho, \frac{1}{\rho} \right\} \sum_{k=1}^{n-1} \epsilon^{m_k} \Im \langle B A(\omega)^{k-1} x, B A(\omega)^k x \rangle. \tag{32}$$

Thanks to (29), one has

$$\begin{aligned} \frac{1}{2} |x|^2 &\leq \mathcal{L}_{\rho, \omega, \epsilon}(x) \leq \frac{3}{2} |x|^2, \quad \forall x \in \mathbb{C}^n, \\ \forall \rho \in (0, +\infty), \quad \forall \omega \in S^{m-1}, \quad \forall \epsilon \in (0, \epsilon_*). \end{aligned} \tag{33}$$

Let $\epsilon \in (0, \epsilon_*)$. In order to simplify the notation and since ϵ is fixed, in the sequel we will write $N_*(\omega)$ and $\mathcal{L}_{\rho, \omega}$ instead of $N_{*, \epsilon}(\omega)$ and $\mathcal{L}_{\rho, \omega, \epsilon}$.

When x solves (20), one has

$$\begin{aligned} \frac{d}{dt} [\mathcal{L}_{\rho, \omega}(x)] &= -2\Re(\langle (B + i\rho A_\omega)x, x \rangle) \\ &\quad - \min \left\{ \rho, \frac{1}{\rho} \right\} \sum_{k=1}^{n-1} \epsilon^{m_k} \Im(\langle B A_\omega^{k-1} (B + i\rho A_\omega)x, B A_\omega^k x \rangle) \\ &\quad - \min \left\{ \rho, \frac{1}{\rho} \right\} \sum_{k=1}^{n-1} \epsilon^{m_k} \Im(B A_\omega^{k-1} x, B A_\omega^k (B + i\rho A_\omega)x) \end{aligned} \tag{34}$$

where $A_\omega := A(\omega)$.

We now distinguish two cases: $0 < \rho < 1$ and $\rho > 1$.

The case $\rho \in (0, 1)$: Using (26), we get

$$\begin{aligned} \frac{d}{dt} [\mathcal{L}_{\rho, \omega}(x)] &\leq -2C_1 |Bx|^2 - \rho^2 \sum_{k=1}^{n-1} \epsilon^{m_k} |B A_\omega^k x|^2 \\ &\quad + \rho \sum_{k=1}^{n-1} \epsilon^{m_k} |Bx| [\|B A_\omega^{k-1}\| \|B A_\omega^k x\| + \|B A_\omega^k\| \|B A_\omega^{k-1} x\|] \\ &\quad + \rho^2 \sum_{k=1}^{n-1} \epsilon^{m_k} |B A_\omega^{k-1} x| |B A_\omega^{k+1} x|. \end{aligned} \tag{35}$$

Moreover, by (27) and (21) we get

$$\begin{aligned}
& \rho \sum_{k=1}^{n-1} \epsilon^{mk} |Bx| [\|BA_{\omega}^{k-1}\| |BA_{\omega}^k x| + \|BA_{\omega}^k\| |BA_{\omega}^{k-1} x|] \\
& \leq \mathcal{M}_1 \rho \sum_{k=1}^{n-1} \epsilon^{mk} |Bx| [|BA_{\omega}^k x| + |BA_{\omega}^{k-1} x|] \\
& \leq \mathcal{M}_1 \rho \epsilon^{m_1} |Bx|^2 + \sum_{k=1}^{n-1} 2\mathcal{M}_1 \rho \epsilon^{mk} |Bx| |BA_{\omega}^k x| \\
& \leq \mathcal{M}_1 \rho \epsilon^{m_1} |Bx|^2 + \sum_{k=1}^{n-1} \left(\frac{C_1}{n-1} |Bx|^2 + \rho^2 \frac{\mathcal{M}_1^2 (n-1)}{C_1} \epsilon^{2mk} |BA_{\omega}^k x|^2 \right) \\
& \leq (C_1 + \mathcal{M}_1 \rho \epsilon^{m_1}) |Bx|^2 + \rho^2 \frac{\mathcal{M}_1^2 (n-1)}{C_1} \sum_{k=1}^{n-1} \epsilon^{2mk} |BA_{\omega}^k x|^2. \tag{36}
\end{aligned}$$

Using (22), we get

$$\begin{aligned}
& \rho^2 \sum_{k=1}^{n-1} \epsilon^{mk} |BA_{\omega}^{k-1} x| |BA_{\omega}^{k+1} x| \\
& \leq \rho^2 \epsilon^{\delta} \sum_{k=1}^{n-1} \epsilon^{\frac{m_{k-1} + m_{k+1}}{2}} |BA_{\omega}^{k-1} x| |BA_{\omega}^{k+1} x| \\
& \leq \frac{1}{2} \rho^2 \epsilon^{\delta} \sum_{k=1}^{n-1} \epsilon^{m_{k-1}} |BA_{\omega}^{k-1} x|^2 + \epsilon^{m_{k+1}} |BA_{\omega}^{k+1} x|^2 \\
& \leq \rho^2 \epsilon^{\delta} \sum_{k=0}^n \epsilon^{mk} |BA_{\omega}^k x|^2. \tag{37}
\end{aligned}$$

Thanks to Cayley–Hamilton theorem, Cauchy–Schwarz inequality, (28) and (21), the last term of the right-hand side in the previous inequality satisfies

$$\begin{aligned}
\rho^2 \epsilon^{\delta + m_n} |BA_{\omega}^n x|^2 &= \rho^2 \epsilon^{\delta + m_n} \left| \sum_{k=0}^{n-1} a_k(\omega) BA_{\omega}^k x \right|^2 \\
&\leq n \mathcal{M}_2^2 \rho^2 \epsilon^{\delta + m_n} \sum_{k=0}^{n-1} |BA_{\omega}^k x|^2 \\
&\leq n \mathcal{M}_2^2 \rho^2 \epsilon^{\delta} \sum_{k=0}^{n-1} \epsilon^{mk} |BA_{\omega}^k x|^2. \tag{38}
\end{aligned}$$

Finally, using (36), (37) and (38) in (35) and thanks to (30), (31), we get

$$\frac{d}{dt} [\mathcal{L}_{\rho, \omega}(x)] \leq -\frac{C_1}{2} |Bx|^2 - \frac{\rho^2}{2} \sum_{k=1}^{n-1} \epsilon^{mk} |BA_{\omega}^k x|^2.$$

Therefore, using (24) and (33) we get

$$\frac{d}{dt} [\mathcal{L}_{\rho,\omega}(x(t))] \leq -2\tilde{c}N_*(\omega)\rho^2\mathcal{L}_{\rho,\omega}(x(t)) \tag{39}$$

where

$$\tilde{c} := \frac{1}{6} \min\{C_1, 1\}.$$

Finally, thanks to (33), we get

$$|x(t)| \leq \sqrt{3}|x_0|e^{-\tilde{c}N_*(\omega)\rho^2t}.$$

The case $\rho \in (1, +\infty)$: Now, let us justify the decay estimate in Proposition 1 for $\rho \in (1, +\infty)$. Let $\rho \in (1, +\infty)$ and $\omega \in S^{m-1}$. When x solves (20), with the same arguments as in the previous case, we get

$$\begin{aligned} \frac{d}{dt} [\mathcal{L}_{\rho,\omega}(x(t))] &\leq -2C_1|Bx|^2 - \sum_{k=1}^{n-1} \epsilon^{m_k} |BA_{\omega}^k x|^2 \\ &\quad + \frac{1}{\rho} \sum_{k=1}^{n-1} \epsilon^{m_k} |Bx| [\|BA_{\omega}^{k-1}\| \|BA_{\omega}^k x\| + \|BA_{\omega}^k\| \|BA_{\omega}^{k-1} x\|] \\ &\quad + \sum_{k=1}^{n-1} \epsilon^{m_k} |BA_{\omega}^{k-1} x| \|BA_{\omega}^{k+1} x\| \end{aligned} \tag{40}$$

and

$$\frac{d}{dt} [\mathcal{L}_{\rho,\omega}(x(t))] \leq -\frac{C_1}{2}|Bx|^2 - \frac{1}{2} \sum_{k=1}^{n-1} \epsilon^{m_k} |BA_{\omega}^k x|^2.$$

Note that the estimates we get are very close to those of the case $\rho \in (0, 1)$ except for the fact that, due to the weight $1/\rho$ on the second term in (32), all terms in which ρ appears have to be divided by ρ .

Finally, with the same computations as in the first case, we get

$$|x(t)| \leq \sqrt{3}|x_0|e^{-\tilde{c}N_*(\omega)t}.$$

□

Remark 4. The explicit Lyapunov function (32) is inspired by those introduced by Villani [34,35] to derive decay estimates for partially diffusive systems. In [35], the operators under consideration are of the form $\mathcal{L} = \mathcal{A}^*\mathcal{A} + \mathcal{B}$, where \mathcal{B} is anti-symmetric. In our case the operator has, rather, the form $L = B + i\rho A(\omega)$ where $i\rho A(\omega)$ is antisymmetric but B does not necessarily have any symmetry property.

The Lyapunov functional we use is also similar to those used in the analysis of the decay properties of dissipative wave equations as, for instance, $v_{tt} - \Delta v + v_t = 0$. There, the systems under consideration are second order (in time), and in the Fourier

setting they take the form of the dissipated harmonic oscillator $\hat{x}'' + |\xi|^2 \hat{x} + \hat{x}' = 0$. The energy of the system is then given by

$$e(t) = \frac{1}{2} \left[|\hat{x}|^2 + |\hat{x}'|^2 \right],$$

while the Lyapunov functional to be used to derive the decay is of the form

$$\mathcal{L}(t) = \frac{1}{2} \left[|\hat{x}|^2 + |\hat{x}'|^2 \right] + \varepsilon \hat{x} \hat{x}'.$$

This corresponds precisely to functionals of the form (32) in the particular case in which $n = 2$.

2.3. A new proof for Shizuta and Kawashima's decomposition

As a consequence of Proposition 1, the decomposition (14) is straightforward.

Theorem 1. *We assume that B has form (5), A_1, \dots, A_m are symmetric and (SK) is satisfied. Then, there exist*

$$C = C(A_1, \dots, A_m, B), \quad \lambda = \lambda(A_1, \dots, A_m, B) > 0$$

such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n)$, the solution $w(t, x)$ of (4) can be decomposed as in (13) where (14) holds.

Proof of Theorem 1. Let $\epsilon_* > 0$ be as in Proposition 1 and $\epsilon \in (0, \epsilon_*)$. Thanks to the implication “(1) \Rightarrow (4)” in Lemma 1, and in view of the (SK) assumption, we have $N_{*,\epsilon}(\omega) > 0$ for every $\omega \in S^{m-1}$. Moreover, the function $\omega \mapsto N_{*,\epsilon}(\omega)$ is continuous on the compact set S^{m-1} and, therefore, there exists $N_* > 0$ such that $N_{*,\epsilon}(\omega) \geq N_*$, for every $\omega \in S^{m-1}$. We define w_1 and w_2 by

$$\hat{w}_1(t, \xi) := \hat{w}(t, \xi) 1_{|\xi| > 1} \quad \text{and} \quad \hat{w}_2(t, \xi) := \hat{w}(t, \xi) 1_{|\xi| < 1}.$$

Thanks to Proposition 1, one has,

$$|\hat{w}_1(t, \xi)| \leq \sqrt{3} |\hat{w}^0(\xi)| e^{-\tilde{c} N_* t}, \quad |\hat{w}_2(t, \xi)| \leq \sqrt{3} |\hat{w}^0(\xi)| e^{-\tilde{c} N_* |\xi|^2 t},$$

for all $\xi \in \mathbb{R}^m$ and $t \in (0, +\infty)$. Thus, one has

$$\begin{aligned} \|w_1(t)\|_{L^2(\mathbb{R}^m, \mathbb{R}^n)} &\leq \sqrt{3} \|w^0\|_{L^2(\mathbb{R}^m, \mathbb{R}^n)} e^{-\tilde{c} N_* t}, \\ \|w_2(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} &\leq \|\hat{w}_2(t)\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)} \\ &\leq \sqrt{3} \int_{\mathbb{R}^m} |\hat{w}^0(\xi)| e^{-\tilde{c} N_* |\xi|^2 t} d\xi \\ &\leq C t^{-m/2} \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}. \end{aligned}$$

□

Remark 5. Let us compare the tools developed here with those in [30]. In [30], the authors use algebraic tools to justify the equivalence between (SK) and the existence of a compensating function (notation defined below) for (4). Then, they use this compensating function to prove the decomposition of Proposition 1 with an energy approach.

Definition 1. A C^∞ map $\omega \in S^{m-1} \mapsto K(\omega) \in \mathbb{C}^{n \times n}$ is a compensating function for (4) if

- $K(-\omega) = -K(\omega), \forall \omega \in S^{m-1}$,
- $K(\omega)$ is a skew-symmetric matrix, for every $\omega \in S^{m-1}$,
- $\frac{B^*+B}{2} + \frac{1}{2}(K(\omega)A(\omega) + (K(\omega)A(\omega))^t)$ is positive definite for every $\omega \in S^{m-1}$.

The proof of Proposition 1 contains arguments to justify that, when $\epsilon^\delta(1 + n\mathcal{M}_2^2) < 1/2$, the expression

$$K(\omega) := \sum_{k=1}^{n-1} \epsilon^{mk} [(A(\omega)^t)^k B^t B A(\omega)^{k-1} - (A(\omega)^t)^{k-1} B^t B A(\omega)^k]$$

defines a compensating function and that

$$\frac{B^* + B}{2} + \frac{1}{2}(K(\omega)A(\omega) + (K(\omega)A(\omega))^*) \geq \frac{1}{2} \min\{C_1, 1\} N_{*,\epsilon}(\omega).$$

Notice that once a compensating function $K(\omega)$ is known, the expression

$$\tilde{\mathcal{L}}_\epsilon(x) := |x|^2 + \epsilon \min\{1, \rho\} \Im \langle K(\omega)x, x \rangle$$

for $\epsilon > 0$ small enough, provides a Lyapunov function for the proof of Proposition 1.

Our proof, however, is much more direct and yields the desired decay rate by means of an explicit Lyapunov function, inspired by Lemma 1.

3. L^2 -stability and nondissipated solutions

This section is devoted to the study of L^2 -stability and the nondissipated solutions of (4) when (SK) does not hold, that is $\mathcal{D}(A_1, \dots, A_m, B) \neq \{0\}$.

In Section 3.1, we show that there are many situations in which (SK) does not hold, which is a motivation for the analysis developed in the following subsections. In Section 3.2 we prove that either $\mathcal{D}(A_1, \dots, A_m, B)$ is a strict algebraic submanifold of \mathbb{R}^m , or $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$. In Section 3.3, we prove that, in the first case, (4) is strongly stable in L^2 , that is, all L^2 solutions tend to zero in L^2 as $t \rightarrow \infty$. In Section 3.4, we prove that, in the second case, there exist non-dissipated solutions of constant L^2 -norm. In Section 3.5, we deduce a complete classification of the decomposition of the solutions in one dimension ($m = 1$).

3.1. (SK) may not be satisfied

The goal of this section is to emphasize that situations exist in which (SK) does not hold. This fact motivates the analysis developed in the following subsections.

The first case is when $m > n$ and B has form (5) with $n_2 < n$. Indeed, in that case the set

$$\left\{ \xi \in \mathbb{R}^m; \sum_{j=1}^m a_{k,1}^{(j)} \xi_j = 0 \text{ for } k = 2, \dots, n \right\}$$

is a nonempty vector subspace of \mathbb{R}^m . It is contained in $\mathcal{D}(A_1, \dots, A_m, B)$. Indeed, for every ξ in this set, we have $A(\xi)e_1 = (\sum_{j=1}^m a_{1,1}^{(j)} \xi_j)e_1$, thus e_1 is an eigenvector of $A(\xi)$ that belongs to $\text{Ker}(B)$.

As a consequence, the property (SK) may be satisfied only when $m \leq n$. However, for any pair (m, n) with $m \leq n$, and any value of n_1 , there are examples for which (SK) does not hold. Indeed, let $m, n \in \mathbb{N}^*$ with $m \leq n, n_1 \in \{1, \dots, n-1\}$ and $n * n$ real matrices A_1, \dots, A_m, B with A_1, \dots, A_m symmetric and B of the form (5). We assume that $a_{2,1}^{(1)} \neq 0, a_{2,1}^{(j)} = 0$ for $j = 2, \dots, m$, and $a_{k,1}^{(j)} = 0$ for $j = 1, \dots, m$ and $k = 3, \dots, n$. Then $A(\xi)e_1 = (\sum_{j=1}^m a_{1,1}^{(j)} \xi_j)e_1 + a_{2,1}^{(1)} \xi_1 e_2$. Thus, for every ξ in the hyperplane

$$\{\xi \in \mathbb{R}^m; \xi_1 = 0\},$$

e_1 is an eigenvector of $A(\xi)$ that belongs to $\text{Ker}(B)$. Therefore, $\mathcal{D}(A_1, \dots, A_m, B)$ contains this hyperplane and (SK) is not fulfilled.

3.2. The set of degeneracy

Proposition 2. *Let A_1, \dots, A_m, B be $n * n$ real matrices such that A_1, \dots, A_m are symmetric and B has form (5). The set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ is an algebraic submanifold of \mathbb{R}^m . In other words, there exists a finite family of polynomials $(P_j)_{j \in J} \subset \mathbb{R}[X]$ such that*

$$\mathcal{D}(A_1, \dots, A_m, B) = \{\xi \in \mathbb{R}^m; P_j(\xi) = 0, \forall j \in J\}. \tag{41}$$

Thus, either $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$, or $\mathcal{D}(A_1, \dots, A_m, B)$ has zero measure. Moreover, $\mathcal{D}(A_1, \dots, A_m, B)$ is stable by homotheties.

Remark 6. Note that it can happen that $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$, the dissipation matrix B being nontrivial. Indeed, the fact that $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$ means only that, for each ξ , there is a direction in which the dissipation mechanism is not effective; but this is compatible, with B being nontrivial.

Proof of Proposition 2. Thanks to Lemma 1, one may characterize the set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ as the set of those $\xi \in \mathbb{R}^m$ such that any $n * n$ subdeterminant of the Kalman matrix K defined in (18) (with A replaced by $A(\xi)$) vanishes. The polynomials P_j in the statement of Proposition 2 are precisely those corresponding to these subdeterminants.

Then, either all the polynomials vanish identically, and then

$$\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m,$$

or, if some of them do not, the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes.

With the definition (15) of the set of degeneracy, it is clear that, when $\xi \in \mathcal{D}(A_1, \dots, A_m, B)$ and $\alpha \in \mathbb{R}$, then, $\alpha\xi \in \mathcal{D}(A_1, \dots, A_m, B)$. \square

According to the previous propositions there are, therefore, only two possible cases, depending on the size of the set of degeneracy. Of course, the smaller $\mathcal{D}(A_1, \dots, A_m, B)$ is, better decay properties we expect. The case in which (SK) holds is a very particular instance where the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes since, in that case, actually, $\mathcal{D}(A_1, \dots, A_m, B)$ is reduced to the trivial set $\{0\}$. One of the main goals of this section is to describe what happens when the measure of the set of degeneracy vanishes but (SK) does not hold.

To that end, it is first convenient to consider some examples. The first conclusion is that, in some cases, $\mathcal{D}(A_1, \dots, A_m, B)$ is a vector subspace, but this is not necessarily always the case. This makes the analysis of the decay rate of (7) as a function of ξ more delicate.

We introduce the notation

$$\alpha_{k,l} = \alpha_{k,l}(\xi) := \sum_{j=1}^m a_{k,l}^{(j)} \xi_j,$$

for $\xi \in \mathbb{R}^m$, $1 \leq k, l \leq n$, where $A_j = (a_{k,l}^{(j)})_{1 \leq k, l \leq n}$. Then $A(\xi) = (\alpha_{k,l})_{1 \leq k, l \leq n}$. Obviously, $\alpha_{k,l} = \alpha_{l,k}$, $1 \leq k, l \leq n$. In the following examples, it is easier to use the definition (15) of $\mathcal{D}(A_1, \dots, A_m, B)$ in order to compute the polynomials than using the minors of the Kalman matrix.

Example 1. Let us consider the case $n_1 = 1$. The definition gives

$$\mathcal{D}(A_1, \dots, A_m, B) = \{\xi \in \mathbb{R}^m; \alpha_{k,1} = 0, \forall k \in \{2, \dots, n\}\}.$$

Thus, $\mathcal{D}(A_1, \dots, A_m, B)$ is a vector subspace.

Example 2. Let us consider the case $n_1 = 2$. The definition allows characterizing $\mathcal{D}(A_1, \dots, A_m, B)$ as the set of those $\xi \in \mathbb{R}^m$ for which there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{1,2} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad \alpha\alpha_{k,1} + \beta\alpha_{k,2} = 0, \quad \forall k \geq 3.$$

Computing explicitly α, β as functions of $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2}$, distinguishing the cases $(\alpha, \beta) = (1, 0)$, $(\alpha, \beta) = (0, 1)$ and $(\alpha \neq 0$ and $\beta \neq 0)$, we get

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_{3+} \cup \mathcal{D}_{3-},$$

where

$$\mathcal{D}_j(A_1, \dots, A_m, B) := \{\xi \in \mathbb{R}^m; \alpha_{k,j} = 0, \forall k \neq j\} \quad \text{for } j \in \{1, 2\},$$

and $\mathcal{D}_{3\pm}(A_1, \dots, A_m, B)$ are the sets of those $\xi \in \mathbb{R}^m$ such that, for every $k \geq 3$,

$$2\alpha_{1,2}\alpha_{k,1} = \left(\alpha_{1,1} - \alpha_{2,2} \pm \sqrt{(\alpha_{1,1} - \alpha_{2,2})^2 + 4\alpha_{1,2}^2} \right) \alpha_{k,2}.$$

Now, we distinguish two cases.

Example 2.a. Let us consider matrices A_1, \dots, A_m such that $a_{1,1}^{(j)} - a_{2,2}^{(j)} = \delta a_{1,2}^{(j)}$ for $j = 1, \dots, m$, for some $\delta \in \mathbb{R}^*$. Then $\alpha_{1,1} - \alpha_{2,2} = \delta \alpha_{1,2}$ and

$$\mathcal{D}_{3+} \cup \mathcal{D}_{3-}(A_1, \dots, A_m, B) = \{ \xi \in \mathbb{R}^m; 2\alpha_{k,1} = [\delta \pm \sqrt{\delta^2 + 4}] \alpha_{k,2}, \forall 3 \leq k \leq n \}.$$

Thus $\mathcal{D}(A_1, \dots, A_m, B)$ is the union of a finite number of vector subspaces of \mathbb{R}^m .

Example 2.b. Now, let us take $n = 3, m = 5$ and A_1, \dots, A_5 defined by

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix},$$

$$A_4 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & * \end{pmatrix}, \quad A_5 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & * \end{pmatrix},$$

where the notation $*$ means that the corresponding coefficient may have any real value. Then, $\alpha_{1,1} = \xi_1, \alpha_{2,2} = \xi_2, \alpha_{1,2} = \xi_3, \alpha_{3,1} = \xi_4, \alpha_{3,2} = \xi_5$. Thus

$$\mathcal{D}_{3\pm}(A_1, \dots, A_5, B) = \{ \xi \in \mathbb{R}^5; 2\xi_3\xi_4 + (\xi_2 - \xi_1)\xi_5 = \pm \xi_5 \sqrt{(\xi_1 - \xi_2)^2 + 4\xi_3^2} \}.$$

Obviously, $\mathcal{D}_{3\pm}$ are not vector subspaces and, consequently, the set of degeneracy can not be written as the union of a finite number of vector subspaces of \mathbb{R}^m .

3.3. An NSC for strong L^2 stability

Proposition 3. *Let A_1, \dots, A_m, B be $n * n$ real matrices with B of the form (5), A_1, \dots, A_m being symmetric. The following statements are equivalent*

- (1) *the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes,*
- (2) *any solution of (4) with $L^2(\mathbb{R}^m, \mathbb{R}^n)$ initial condition converges to zero strongly in $L^2(\mathbb{R}^m, \mathbb{R}^n)$ as $t \rightarrow \infty$.*

Proof. First, we prove that “(1) \Rightarrow (2)”. We assume that the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes. Let $w_0 \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ and w be the solution of (4) with initial condition w_0 . Applying Proposition 1, we get, for $\epsilon > 0$ small enough,

$$|\hat{w}(t, \xi)| \leq \sqrt{3} |\hat{w}_0(\xi)| e^{-\tilde{c} \min\{|\xi|^2, 1\} N_{*,\epsilon}(\omega)t}$$

where $\omega := \xi/|\xi|$. Thanks to the implication “(1) \Rightarrow (4)” in Lemma 1 and since the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes, then, for almost every $\omega \in S^{m-1}$,

$N_{*,\epsilon}(\omega) > 0$. Thus $\hat{w}(t, \xi) \rightarrow 0$ when $t \rightarrow +\infty$ for almost every $\xi \in \mathbb{R}^m$. Moreover, $|\hat{w}(t, \cdot)| \leq \sqrt{3}|\hat{w}_0(\cdot)| \in L^2(\mathbb{R}^m, \mathbb{R}^n)$. Thus, the dominated convergence theorem implies that $\hat{w}(t)$ converges to zero in $L^2(\mathbb{R}^m, \mathbb{R}^n)$ when $t \rightarrow +\infty$.

Now, we prove “(2) \Rightarrow (1)”. Let us assume (2) holds. Let $\varphi \in \mathcal{S}(\mathbb{R}^m, \mathbb{R})$ be such that $\varphi > 0$ on \mathbb{R}^m . Let $w_0^{(1)} \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ be defined by $\hat{w}_0^{(1)}(\xi) := \varphi(\xi)e_1$ and $w^{(1)}$ be the solution of (4) with initial condition $w_0^{(1)}$. One has

$$\hat{w}^{(1)}(t, \xi) = \exp[E(\xi)t]e_1\varphi(\xi).$$

Since $w^{(1)}(t) \rightarrow 0$ strongly in $L^2(\mathbb{R}^m, \mathbb{R}^n)$, there exists an increasing sequence $(t_p^{(1)})_{p \in \mathbb{N}} \subset \mathbb{R}_+$ such that

$$\exp[E(\xi)t_p^{(1)}]e_1 \rightarrow 0 \text{ when } [p \rightarrow +\infty] \text{ for almost every } \xi \in \mathbb{R}^m.$$

Let $w_0^{(2)} \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ be defined by $\hat{w}_0^{(2)}(\xi) := \varphi(\xi)e_2$ and $w^{(2)}$ be the solution of (4) with initial condition $w_0^{(2)}$. The sequence $(w^{(2)}(t_p^{(1)}))_{p \in \mathbb{N}}$ tends to zero strongly in $L^2(\mathbb{R}^m, \mathbb{R}^n)$. Thus, there exists a subsequence $(t_p^{(2)})_{p \in \mathbb{N}}$ of $(t_p^{(1)})_{p \in \mathbb{N}}$ such that

$$\exp[E(\xi)t_p^{(2)}]e_2 \rightarrow 0 \text{ when } [p \rightarrow +\infty] \text{ for almost every } \xi \in \mathbb{R}^m.$$

Iterating this process for $k = 1, \dots, n$, one gets an increasing sequence $(t_p)_{p \in \mathbb{N}} \subset \mathbb{R}_+$ and a subset \mathcal{N} of \mathbb{R}^m of zero measure such that

$$\exp[E(\xi)t_p] \rightarrow 0 \text{ in } \mathbb{R}^{n \times n}, \quad \forall \xi \in \mathbb{R}^m - \mathcal{N}. \tag{42}$$

Let $\xi \in \mathbb{R}^m - \mathcal{N}$. The convergence (42) implies that any eigenvalue of $E(\xi)$ has a negative real part. Thus, $A(\xi)$ has no eigenvector in $\text{Ker}(B)$ since, otherwise, one would get an eigenvalue λ of $E(\xi)$ with $\Re(\lambda) = 0$. In conclusion, $\mathcal{D}(A_1, \dots, A_m, B)$ is contained in \mathcal{N} and, consequently, its measure vanishes. \square

Remark 7. As a consequence of this result, and in view of Proposition 2 which classifies the possible structures for the set of degeneracy, we deduce that, whenever $\mathcal{D}(A_1, \dots, A_m, B)$ is a strict subset of \mathbb{R}^m , all solutions tend to zero in L^2 strongly as $t \rightarrow \infty$. In the following section we show that, in the other case, that is, when $\mathcal{D}(A_1, \dots, A_m, B)$ coincides with \mathbb{R}^m , there are nontrivial solutions of constant L^2 -norm.

3.4. An NSC for the existence of nondissipated solutions

Proposition 4. *Let A_1, \dots, A_m, B be $n \times n$ real matrices with B of the form (5) and A_1, \dots, A_m symmetric. The following statements are equivalent*

- (1) $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$,
- (2) *there exists a nondissipated solution of (4), that is with a constant L^2 -norm.*

The following Lemma will be important in the proof of Proposition 4.

Lemma 2. Let A_1, \dots, A_m, B be $n * n$ real matrices with B of the form (5) and A_1, \dots, A_m symmetric. Let $w^0 \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ and w be the solution of (4) with initial condition w^0 . The following statements are equivalent

- (1) w is not dissipated, that is $\|w(t)\|_{L^2} \equiv \|w^0\|_{L^2}$,
- (2) for almost every $\xi \in \mathbb{R}^m$, $\hat{w}^0(\xi)$ belongs to

$$V(\xi) := \{v \in \mathbb{C}^n, BA(\xi)^k v = 0, \forall k \in \mathbb{N}\}.$$

Proof of Lemma 2. Let us prove (1) \Rightarrow (2). We assume w is not dissipated. We have

$$\frac{d}{dt} \int_{\mathbb{R}^m} |w(t, x)|^2 dx = -2 \int_{\mathbb{R}^m} \langle Bw(t, x), w(t, x) \rangle dx \equiv 0,$$

thus $Bw \equiv 0$ (see the special structure of B), which implies $B\hat{w} \equiv 0$. Thus the expression (9) can be written

$$\hat{w}(t, \xi) = \exp[-iA(\xi)t]\hat{w}^0(\xi).$$

The equality $B\hat{w} \equiv 0$ implies

$$B \exp[-iA(\xi)t]\hat{w}^0(\xi) = 0, \text{ for every } t \in \mathbb{R}_+, \text{ for almost every } \xi \in \mathbb{R}^m.$$

Differentiating k times this equality with respect to time at $t = 0$, we get (2).

Now, let us prove (2) \Rightarrow (1). We assume that, for almost every $\xi \in \mathbb{R}^m$, $\hat{w}_0(\xi) \in V(\xi)$. By definition, $V(\xi)$ is a vector subspace of $\text{Ker}(B)$ stable by $A(\xi)$, thus (9) gives

$$\begin{aligned} \hat{w}(t, \xi) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} [-B - iA(\xi)]^k \hat{w}^0(\xi) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} [-iA(\xi)]^k \hat{w}^0(\xi) \\ &= \exp[-iA(\xi)t]\hat{w}^0(\xi). \end{aligned}$$

Therefore $\|w(t)\|_{L^2} = \|\hat{w}(t)\|_{L^2} \equiv \|w_0\|_{L^2}$, that is, (1) holds. \square

Proof of Proposition 4. Let us prove (2) \Rightarrow (1). We assume there exists a nondissipated solution w , associated to an initial condition $w^0 \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ with $w^0 \neq 0$. Thanks to Lemma 2, we know that, for almost every $\xi \in \mathbb{R}^m$, $\hat{w}^0(\xi) \in V(\xi)$. So, for almost every $\xi \in \text{Supp}(w_0)$ (the support of w_0), $V(\xi)$ is a nonempty vector subspace of $\text{Ker}(B)$ stable by $A(\xi)$; moreover $A(\xi)|_{V(\xi)}$ is symmetric, thus it has an eigenvector. We have proved that $\text{Supp}(w^0) \subset \mathcal{D}(A_1, \dots, A_m, B)$, thus the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ does not vanish and Proposition 2 implies that (1) holds.

Now, let us prove (1) \Rightarrow (2). We assume $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$. Then for every $\xi \in \mathbb{R}^m$, $V(\xi) \neq \{0\}$ because $V(\xi)$ contains at least an eigenvector of $A(\xi)$ that belongs to $\text{Ker}(B)$. In order to prove that (2) holds, we build an $L^2(\mathbb{R}^m, \mathbb{R}^n)$

initial condition w^0 such that $\hat{w}^0(\xi) \in V(\xi)$, for every $\xi \in \mathbb{R}^m$, which gives the conclusion, thanks to Lemma 2.

The Cayley–Hamilton theorem justifies that

$$V(\xi) = \{v \in \mathbb{C}^n; BA(\xi)^k v = 0 \text{ for } k = 0, \dots, n - 1\} = \text{Ker}(M_1(\xi)),$$

where $M_1(\xi) = \sum_{k=0}^{n-1} A(\xi)^k B^t BA(\xi)^k$ depends polynomially on ξ . For $\xi \in \mathbb{R}^m$, let $P(\xi)$ be the orthogonal projection $\mathbb{R}^n \rightarrow \text{Ker}(M_1(\xi))$. Let $\xi^* \in \mathbb{R}^m$ and $z^* \in S^{n-1} \cap \text{Ker}(M_1(\xi^*))$. The map $\xi \in \mathbb{R}^m \mapsto P(\xi)z^*$ is continuous at ξ^* because $M_1(\xi)$ is an analytic perturbation of $M_1(\xi^*)$ (see, for example [21, Chapter 2, Section 1.4]). Thus, there exists a neighborhood Ω_1 of ξ^* in \mathbb{R}^m with a positive finite measure such that $\|P(\xi)z^*\| > 1/2$, for every $\xi \in \Omega_1$. We define

$$\hat{w}^0(\xi) := P(\xi)z^* 1_{\Omega_1 \cup (-\Omega_1)}(\xi).$$

Then $w^0 \in L^2(\mathbb{R}^m, \mathbb{R}^n)$ because $|\hat{w}^0(\xi)| \leq 1_{\Omega_1 \cup (-\Omega_1)}(\xi)$ and $\hat{w}^0(-\xi) = \overline{\hat{w}^0(\xi)}$. Moreover, $w^0 \neq 0$ because $\|\hat{w}^0(\xi)\| \geq 1/2, \forall \xi \in \Omega_1 \cup (-\Omega_1)$. \square

Remark 8. According to the results in this section and the previous one, we deduce that there are only two possibilities for the set of degeneracy which lead to different asymptotic properties as $t \rightarrow \infty$:

- Whenever $\mathcal{D}(A_1, \dots, A_m, B)$ is a strict subset of \mathbb{R}^m , its measure vanishes, and all solutions tend to zero in L^2 strongly as $t \rightarrow \infty$.
- When $\mathcal{D}(A_1, \dots, A_m, B)$ coincides with \mathbb{R}^m , there are nontrivial solutions of constant L^2 -norm.

3.5. Complete classification in the case $m = 1$

In this section we consider the one-dimensional problem

$$w_t + Aw_x = Bw, \tag{43}$$

where $A, B \in \mathbb{R}^{n \times n}$, $w \in \mathbb{R}^n$ and $x \in \mathbb{R}$. Note that this simple case can be treated directly and without the analysis developed in the previous subsections. However, it is clarifying to treat it in the framework of the general theory we have developed.

When $m = 1$, $A(\xi) = \xi A$, since ξ is now scalar. Therefore, the (SK) condition does not involve the Fourier variable ξ and can be written simply as

$$\text{(SK): } \text{Ker}(B) \cap \{\text{eigenvectors of } A\} = \{0\}.$$

Obviously, this is equivalent to requiring that the pair (A, B) satisfy the Kalman rank condition.

In one dimension ($m = 1$), the (SK) condition characterizes completely the asymptotic behavior of the solutions of (43).

Proposition 5. *We assume that A is symmetric and B has the form (5). The following holds:*

- *The following statements are equivalent*

- (1) A and B satisfy (SK): $\{\text{eigenvectors of } A\} \cap \text{Ker}(B) = \{0\}$,
- (2) there exist $C, \lambda > 0$ such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}, \mathbb{R}^n)$, the solution of (43) can be decomposed as $w = w_1 + w_2$ where (14) holds with $m = 1$.
- Moreover, if (SK) is not satisfied, any solution of (43) with initial condition in $L^2(\mathbb{R}, \mathbb{R}^n)$ may be decomposed as $w = w_1 + w_2 + w_{trw}$ where w_1, w_2 satisfy the previous estimates and w_{trw} is a pure transport term constituted by a finite sum of traveling waves, $w_{trw}(t, x) = \sum_{j=1}^r w_{trw,j}(x - \lambda_j t)$, where $r \leq n$, $\lambda_j \in \mathbb{R}$ and $w_{trw,j} \in L^2(\mathbb{R}, \mathbb{R}^n)$ for $j = 0, \dots, r$.

Proof of Proposition 5. First, the implication “(1) \Rightarrow (2)” was proved in Theorem 1.

Now, we prove the second statement, that also gives “(2) \Rightarrow (1)”.

Assume that (SK) is false. Let V be the subspace of $\text{Ker}(B)$ which is stable by A ($AV \subset V$) with maximal dimension constituted of the sum of all subspaces of $\text{Ker}(B)$ stable by A . Let $r := \dim(V)$. The endomorphism $A|_V$ is symmetric, thus there exists an orthonormal basis (v_1, \dots, v_r) of V made of eigenvectors of A : $Av_j = \lambda_j v_j$, $j = 1, \dots, r$. Let V^\perp be the orthogonal supplementary of V in $\text{Ker}(B)$ (that is $\text{Ker}(B) = V + V^\perp$) and $(v_{r+1}, \dots, v_{n_1})$ be an orthonormal basis of V^\perp . Then, (v_1, \dots, v_{n_1}) is an orthonormal basis of $\text{Ker}(B)$. Since $\text{Ker}(B) = \text{Span}(e_1, \dots, e_{n_1})$, then $\mathcal{V} := (v_1, \dots, v_{n_1}, e_{n_1+1}, \dots, e_n)$ is an orthonormal basis of \mathbb{R}^n . Let P be the basis change matrix from the canonical basis (e_1, \dots, e_n) to the basis \mathcal{V} (the columns of P are the components of the vectors of \mathcal{V} in the canonical basis (e_1, \dots, e_n)). Then $P^{-1}BP = B$,

$$\begin{aligned} \langle Av_j, v_i \rangle &= \langle v_j, Av_i \rangle = \lambda_i \langle v_j, v_i \rangle = 0, \quad \forall j \in \{r+1, \dots, n_1\}, \quad \forall i \in \{1, \dots, r\}, \\ \langle Ae_j, v_i \rangle &= \langle e_j, Av_i \rangle = \lambda_i \langle e_j, v_i \rangle = 0, \quad \forall j \in \{n_1+1, \dots, n\}, \quad \forall i \in \{1, \dots, r\}. \end{aligned}$$

Thus

$$\tilde{A} := P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \dots 0 & 0 \dots 0 \\ \dots & \dots & \dots & \dots \\ & \lambda_r & 0 \dots 0 & 0 \dots 0 \\ \hline 0 \dots 0 & * \dots * & * \dots * \\ 0 \dots 0 & * \dots * & * \dots * \\ 0 \dots 0 & * \dots * & * \dots * \\ \hline 0 \dots 0 & * \dots * & * \dots * \\ 0 \dots 0 & * \dots * & * \dots * \\ 0 \dots 0 & * \dots * & * \dots * \end{pmatrix}$$

where the first diagonal block is of dimension $r * r$, the second is of dimension $(n_1 - r) * (n_1 - r)$ and the third is of dimension $n_2 * n_2$. Then $\tilde{w} := P^{-1}w$ solves

$$\tilde{w}_t + \tilde{A}\tilde{w}_x = -B\tilde{w}$$

and $\tilde{w}_j(t, x) = \tilde{w}_j(0, x - \lambda_j t)$ for $j = 1, \dots, r$. These components are pure transport terms. We define

$$w_{trw} := P \begin{pmatrix} \tilde{w}_1 \\ \dots \\ \tilde{w}_r \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

The $(n - r)$ latest components of \tilde{w} solve a hyperbolic system fulfilling the (SK) condition. Thus, it may be decomposed as $\tilde{w}_1 + \tilde{w}_2$ with (14) for $m = 1$. Finally, we take $w_1 := P\tilde{w}_1$ and $w_2 := P\tilde{w}_2$. This completes the decomposition in three terms of the second statement.

Note that this argument shows that “(2) \Rightarrow (1)”, as claimed. Indeed, if (1) is not satisfied, that is, the (SK) is not fulfilled, then the argument above shows that there is a traveling wave component that does not decay, thus showing that (2) does not hold either. \square

4. Decomposition of the solutions when \mathcal{D} is the union of vector subspaces

As proved in previous sections, whenever the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes, all solutions tend to zero in L^2 as $t \rightarrow \infty$. This section is devoted to analyzing the asymptotic behavior in more detail. As emphasized in Remark 1, the asymptotic behavior of the solutions of (4) reduces to the study of the real valued function $\omega \in S^{m-1} \mapsto N_{*,\epsilon}(\omega)$, defined in Proposition 1. That study is the key point of this section.

Throughout this section, we will consider situations in which the set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ has zero measure and is the union of vector subspaces. This assumption is restrictive because, in general, when the measure of $\mathcal{D}(A_1, \dots, A_m, B)$ vanishes, this set is an algebraic submanifold (see Proposition 2), which is not necessarily a union of vector subspaces (see Examples 1, 2.a, 2.b in Section 3.2). However, this assumption holds in many particular examples, as studied in this section.

In Section 4.1, we state a decomposition for the solutions of (4) when $\mathcal{D}(A_1, \dots, A_m, B)$ is a union of vector subspaces and under the additional assumption

$$N_{*,\epsilon}(\omega) \geq c \text{dist}(\omega, S^{m-1} \cap \mathcal{D})^\alpha, \quad \forall \omega \in S^{m-1} \tag{44}$$

for some $\alpha \geq 2, c > 0$ (see Proposition 6).

In Section 4.2, we prove that, in the particular case $n_1 = 1$ (that is $\text{Ker}(B) = \text{Span}(e_1)$), then $\mathcal{D}(A_1, \dots, A_m, B)$ is a vector subspace of \mathbb{R}^m and (44) holds with $\alpha = 2$ (see Proposition 7). This leads to a decomposition of the solutions of (4) with $n_1 = 1$ (see Theorem 2).

In Section 4.3, we consider different explicit examples, in which the set of degeneracy $\mathcal{D}(A_1, \dots, A_m, B)$ is a union of vector subspaces and the smallest exponent in (44) is $\alpha = 2$ or $\alpha = 4$. We deduce a decomposition of the solutions of (4) for these particular cases (see Theorem 3). With these examples, we see that, when $m \geq 2$, there is a whole class of phenomena that do not arise under (SK).

Finally, in Section 4.4, we write a complete classification of the different possible asymptotic behaviors when $n = 2$.

4.1. General statement

For $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$, we define $L(g) : (0, +\infty) \rightarrow (0, +\infty]$,

$$L(g)(\rho) := \sup\{|g(\xi)|; \xi \in \mathbb{R}^m, |\xi| = \rho\}.$$

Proposition 6. *We assume $m \geq 2$. Let A_1, \dots, A_m, B be $n * n$ real matrices with A_1, \dots, A_m symmetric and B of the form (5). Let $\epsilon \in (0, \epsilon_*)$ where ϵ_* is as in Proposition 1. We assume*

– (A1)

$$\mathcal{D}(A_1, \dots, A_m, B) = \cup_{j=1}^J \mathcal{D}_j,$$

where $J \in \mathbb{N}^*$ and \mathcal{D}_j is a vector subspace of \mathbb{R}^m with codimension $r_j \in \mathbb{N}^*$,
 – (A2) there exists $c_j > 0, \alpha_j \geq 2$ for $j = 1, \dots, J$, such that, for every $\omega \in S^{m-1}$ with $\text{dist}(\omega, \mathcal{D}(A_1, \dots, A_m, B)) = \text{dist}(\omega, \mathcal{D}_j)$,

$$N_{*,\epsilon}(\omega) \geq c_j \text{dist}(\omega, \mathcal{D}_j)^{\alpha_j}. \tag{45}$$

Then, there exists $C = C(A_1, \dots, A_m, B)$ and $\lambda = \lambda(A_1, \dots, A_m, B) > 0$ such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n)$ with

$$N_1(w^0) := \int_1^\infty \rho^{m-1} L[\hat{w}^0](\rho) \, d\rho < \infty, \tag{46}$$

the solution of (4) with initial condition w^0 can be decomposed as

$$w = w_1 + w_2 + w_3 + w_4$$

where (14) holds and

$$\|w_3(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{t^{\frac{1}{\alpha}}} N_1(w^0), \quad \forall t \in (0, +\infty), \tag{47}$$

where $\alpha := \max\{\alpha_1, \dots, \alpha_J\}$ and

$$\|w_4(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq C \left(\sum_{j=1}^J \frac{1}{t^{\alpha_j}} \right) \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}, \quad \forall t \in (0, +\infty). \tag{48}$$

Remark 9. The assumption (46) holds, in particular, when w^0 belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^m, \mathbb{R}^n)$. Indeed, in that case, $\hat{w}^0 \in \mathcal{S}(\mathbb{R}^m, \mathbb{R}^n)$. Thus, there exists $C > 0$ such that, for every $\xi \in \mathbb{R}^m$,

$$|\hat{w}^0(\xi)| \leq \frac{C}{|\xi|^{m+1}},$$

which leads to

$$\int_1^\infty \rho^{m-1} L(\hat{w}^0)(\rho) d\rho \leq \int_1^\infty \frac{C}{\rho^2} d\rho.$$

Remark 10. We refer to Examples 1 and 2.a of Section 3.2 for which the assumption (A1) holds with different values for the codimensions r_j . For example, in Example 2.a, any \mathcal{D}_j is defined by $(n-2)$ linear equations, but some of them may be linearly dependent, thus the codimension of \mathcal{D}_j may take any value $r_j \in \{1, \dots, m-1\}$ by varying n and the coefficients of the matrices A_1, \dots, A_m .

Note that the assumption $\alpha_j \geq 2$ is natural. Indeed, let us consider $\omega^* \in S^{m-1} \cap \mathcal{D}(A_1, \dots, A_m, B)$ and let us assume that the map $\omega \mapsto N_{*,\epsilon}(\omega)$ is smooth in a neighborhood of ω^* [this can be justified, for example, when $N_{*,\epsilon}(\omega^*) = 0$ is a simple eigenvalue of the matrix $M_\epsilon(\omega^*)$ defined by (23)]. Since $N_{*,\epsilon}$ is nonnegative and $N_{*,\epsilon}(\omega^*) = 0$, then, necessarily, $dN_{*,\epsilon}(\omega^*) = 0$. Thus, the Taylor formula justifies that, in a neighborhood of ω^* we have

$$N_{*,\epsilon}(\omega) \leq |d^2 N_{*,\epsilon}(\omega^*)| |\omega - \omega^*|^2.$$

Therefore, if (A2) holds, then, necessarily $\alpha_j \geq 2$.

Remark 11. Note that in Proposition 6, we do not assume that (SK) holds. Consequently, in the decomposition of the solution, one has two other terms that decay more slowly with rates $t^{-1/\alpha}$ and $\max\{t^{-r_j/\alpha_j}; 1 \leq j \leq J\}$. Note also that, necessarily, $r^j/\alpha_j < m/2$ because $r_j < m$ and $\alpha \geq 2$ whenever $\mathcal{D}(A_1, \dots, A_m, B)$ is not trivial.

Proof of Proposition 6. In order to simplify the notation in this proof, we write \mathcal{D} instead of $\mathcal{D}(A_1, \dots, A_m, B)$.

Let $d \in (0, 1)$. We define w_1, w_2, w_3, w_4 by the following expressions, for every $t \in (0, +\infty)$ and for every $\xi \in \mathbb{R}^*$,

$$\begin{aligned} \hat{w}_1(t, \xi) &:= \hat{w}(t, \xi) \mathbf{1}_{\rho > 1} \mathbf{1}_{\text{dist}(\omega, \mathcal{D}) > d} \\ \hat{w}_2(t, \xi) &:= \hat{w}(t, \xi) \mathbf{1}_{\rho < 1} \mathbf{1}_{\text{dist}(\omega, \mathcal{D}) > d} \\ w_3 &:= \sum_{j=1}^J w_{3,j} \\ \hat{w}_{3,j}(t, \xi) &:= \hat{w}(t, \xi) \mathbf{1}_{\rho > 1} \mathbf{1}_{\text{dist}(\omega, \mathcal{D}) = \text{dist}(\omega, \mathcal{D}_j) < d} \\ w_4 &:= \sum_{j=1}^J w_{4,j} \\ \hat{w}_{4,j}(t, \xi) &:= \hat{w}(t, \xi) \mathbf{1}_{\rho < 1} \mathbf{1}_{\text{dist}(\omega, \mathcal{D}) = \text{dist}(\omega, \mathcal{D}_j) < d}. \end{aligned}$$

Thanks to Proposition 1 and assumption (A2), there exists $c > 0$ depending on d, c_j, α_j such that, for every $\xi \in \mathbb{R}^m$,

$$\begin{aligned} |\hat{w}_1(t, \xi)| &\leq \sqrt{3}|\hat{w}^0(\xi)|e^{-ct}, \\ |\hat{w}_2(t, \xi)| &\leq \sqrt{3}|\hat{w}^0(\xi)|e^{-c\rho^2t}. \end{aligned}$$

The same computations as in the proof of Theorem 1 lead to

$$\begin{aligned} \|w_1(t)\|_{L^2(\mathbb{R}^m, \mathbb{R}^n)} &\leq \sqrt{3}\|w^0\|_{L^2(\mathbb{R}^m, \mathbb{R}^n)}e^{-ct}, \\ \|w_2(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} &\leq \frac{C}{t^{m/2}}\|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}. \end{aligned}$$

We fix $j \in \{1, \dots, J\}$. In the study of $w_{3,j}$ and $w_{4,j}$, we will use new coordinates on \mathbb{R}^m . We introduce the orthogonal sum

$$\mathbb{R}^m = \mathcal{D}_j^\perp + \mathcal{D}_j : \xi = \xi' + \xi''.$$

We fix an orthonormal basis (b'_1, \dots, b'_{r_j}) of \mathcal{D}_j^\perp and another orthonormal basis $(b''_1, \dots, b''_{m-r_j})$ of \mathcal{D}_j . For $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{R}^m - \{0\}$, let $\rho := |\xi| \in \mathbb{R}_+^*$, $\theta \in [0, \pi/2]$ be such that

$$|\xi'| = \rho \cos(\theta) \text{ that is, } \theta := \arccos\left(\frac{|\xi'|}{|\xi|}\right).$$

Then $\xi' = \rho \cos(\theta)\zeta^*$, $\xi'' = \rho \sin(\theta)\zeta^{**}$, where $\zeta^* \in \mathcal{D}_j^\perp \cap S^{m-1}$, $\zeta^{**} \in \mathcal{D}_j \cap S^{m-1}$. There exists a unique pair (α, β) with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{r_j-1}) \in \Omega_\alpha := [-\pi/2, \pi/2) \times [0, 2\pi)^{r_j-2}$, $\beta = (\beta_1, \beta_2, \dots, \beta_{m-r_j-1}) \in \Omega_\beta := [-\pi/2, \pi/2) \times [0, 2\pi)^{m-r_j-2}$, such that

$$\begin{aligned} \zeta^* &= \cos(\alpha_1) \cos(\alpha_2) \dots \cos(\alpha_{r_j-2}) \cos(\alpha_{r_j-1}) b'_1 \\ &\quad + \cos(\alpha_1) \cos(\alpha_2) \dots \cos(\alpha_{r_j-2}) \sin(\alpha_{r_j-1}) b'_2 \\ &\quad + \cos(\alpha_1) \cos(\alpha_2) \dots \sin(\alpha_{r_j-2}) b'_3 \\ &\quad + \dots \\ &\quad \dots \\ &\quad \cos(\alpha_1) \sin(\alpha_2) b'_{r_j-1} \\ &\quad + \sin(\alpha_1) b'_{r_j}, \\ \zeta^{**} &= \cos(\alpha_1) \cos(\alpha_2) \dots \cos(\alpha_{m-r_j-2}) \cos(\alpha_{m-r_j-1}) b''_1 \\ &\quad + \cos(\alpha_1) \cos(\alpha_2) \dots \cos(\alpha_{m-r_j-2}) \sin(\alpha_{m-r_j-1}) b''_2 \\ &\quad + \cos(\alpha_1) \cos(\alpha_2) \dots \sin(\alpha_{m-r_j-2}) b''_3 \\ &\quad + \dots \\ &\quad \dots \\ &\quad \cos(\alpha_1) \sin(\alpha_2) b''_{m-r_j-1} \\ &\quad + \sin(\alpha_1) b''_{m-r_j}. \end{aligned}$$

In the following arguments, we will use the change of variables $(\xi_1, \dots, \xi_m) \rightarrow (\rho, \theta, \alpha, \beta)$, $(\xi_1, \dots, \xi_m) \rightarrow (\xi', \xi'')$. We have

$$d\xi_1 \dots d\xi_m = d\xi' d\xi''$$

because $\xi = (\xi_1, \dots, \xi_m) \mapsto (\xi', \xi'')$ is a unitary linear transformation. We have

$$d\xi_1 \dots d\xi_m = J(\rho, \theta, \alpha, \beta) \, d\rho d\theta d\alpha d\beta,$$

where J is the Jacobian of the change of variables. Since the dependence of ξ with respect to $\theta, \alpha_1, \dots, \alpha_{r_j-1}, \beta_1, \dots, \beta_{m-r_j-1}$ always involves the functions $\cos(\cdot)$ and $\sin(\cdot)$, there exists $C = C(\mathcal{D}_j) > 0$ such that,

$$|J(\rho, \theta, \alpha, \beta)| \leq C\rho^{m-1} \, d\rho d\theta d\alpha d\beta. \tag{49}$$

Now, let us compute a bound on $w_{3,j}$. For $\xi \in \mathbb{R}^m$, we have the orthogonal decomposition

$$\begin{aligned} \mathbb{R}^m &= \mathcal{D}_j^\perp + \mathcal{D}_j \\ \frac{\xi}{|\xi|} &= \frac{\xi'}{|\xi|} + \frac{\xi''}{|\xi|} \end{aligned}$$

thus

$$\frac{|\xi'|}{|\xi|} = \text{dist}\left(\frac{\xi}{|\xi|}, \mathcal{D}_j\right), \tag{50}$$

therefore

$$\begin{aligned} D_{3,j} &:= \left\{ \xi \in \mathbb{R}^m; |\xi| > 1, \text{dist}\left(\frac{\xi}{|\xi|}, \mathcal{D}\right) = \text{dist}\left(\frac{\xi}{|\xi|}, \mathcal{D}_j\right) < d \right\} \\ &= \left\{ \xi \in \mathbb{R}^m; |\xi| > 1, \frac{|\xi'|}{|\xi|} < d \right\} \\ &= \{ \xi \in \mathbb{R}^m; \rho > 1, \cos(\theta) < d \}. \end{aligned}$$

Using the previous inclusion, Proposition 1, the assumption (45) and finally (50), we get

$$\begin{aligned} \|w_{3,j}(t)\|_{L^\infty} &\leq \|\hat{w}_{3,j}(t)\|_{L^1} \\ &\leq \int_{D_{3,j}} |\hat{w}(t, \xi)| \, d\xi \leq \int_{D_{3,j}} \sqrt{3} |\hat{w}^0(\xi)| e^{-cN_*(\xi/|\xi|)t} \, d\xi \\ &\leq \int_{D_{3,j}} \sqrt{3} |\hat{w}^0(\xi)| e^{-cc_j \text{dist}(\xi/|\xi|, \mathcal{D}_j)^{\alpha_j} t} \, d\xi \\ &\leq \int_{D_{3,j}} \sqrt{3} |\hat{w}^0(\xi)| e^{-cc_j (|\xi'|/|\xi|)^{\alpha_j} t} \, d\xi. \end{aligned}$$

Now, we perform a change of variable $(\xi_1, \dots, \xi_m) \rightarrow (\rho, \theta, \alpha, \beta)$. Thanks to (49), we have

$$\|w_{3,j}(t)\|_{L^\infty} \leq \int_1^\infty \int_{\theta(d)}^{\pi/2} \int_{\Omega_\alpha} \int_{\Omega_\beta} C\sqrt{3}L[\hat{w}^0](\rho) e^{-cc_j \cos(\theta)^{\alpha_j} t} \rho^{m-1} \, d\rho d\theta d\alpha d\beta,$$

where $\theta(d) := \arccos(d)$ (indeed $|\xi'|/|\xi| = \cos(\theta) < d$ in $D_{3,j}$). Therefore, we have

$$\|w_{3,j}(t)\|_{L^\infty} \leq C_1 N_1(w^0) \int_{\theta(d)}^{\pi/2} e^{-cc_j \cos(\theta)^{\alpha_j} t} \, d\theta,$$

where $C_1 := C\sqrt{3}\text{vol}(\Omega_\alpha)\text{vol}(\Omega_\beta)$. Performing the change of variable

$$y = \cos(\theta)t^{1/\alpha_j} \quad \text{that is, } \theta = \arccos\left(\frac{y}{t^{1/\alpha_j}}\right),$$

we get

$$\begin{aligned} \int_{\theta(d)}^{\pi/2} e^{-cc_j \cos(\theta)^{\alpha_j} t} d\theta &= \int_0^{t^{1/\alpha_j}} e^{-cc_j y^{\alpha_j}} \frac{1}{\sqrt{1 - \frac{y^2}{t^{2/\alpha_j}}}} \frac{dy}{t^{1/\alpha_j}} \\ &\leq \frac{1}{t^{1/\alpha_j}} \frac{1}{\sqrt{1-d^2}} \int_0^\infty e^{-cc_j y^{\alpha_j}} dy. \end{aligned}$$

This proves (47) with any constant C such that

$$C \geq \frac{C_1}{\sqrt{1-d^2}} \sum_{j=1}^J \int_0^\infty e^{-cc_j y^{\alpha_j}} dy.$$

Now, let us compute a bound on $w_{4,j}$. With the same strategy, we get

$$\begin{aligned} \|w_{4,j}(t)\|_{L^\infty} &\leq \int_{D_{4,j}} \sqrt{3} |\hat{w}^0(\xi)| e^{-c|\xi|^2 N_*(\xi/|\xi|)t} d\xi \\ &\leq \int_{D_{4,j}} \sqrt{3} |\hat{w}^0(\xi)| e^{-cc_j |\xi|^2 (|\xi'|/|\xi|)^{\alpha_j} t} d\xi, \end{aligned}$$

where

$$D_{4,j} := \{\xi \in \mathbb{R}^m; |\xi| < 1, \text{dist}(\xi/|\xi|, \mathcal{D}) = \text{dist}(\xi/|\xi|, \mathcal{D}_j) < d\}.$$

Using $\|\hat{w}^0\|_{L^\infty} \leq \|w^0\|_{L^1}$, the inequality $|\xi|^2 \cos(\theta)^{\alpha_j} \geq [|\xi| \cos(\theta)]^{\alpha_j} = |\xi'|^{\alpha_j}$ (because $\alpha_j \geq 2$) and the inclusion

$$D_{4,j} \subset \{\xi \in \mathbb{R}^m; |\xi'| < d, |\xi''| < 1\},$$

we get

$$\|w_{4,j}(t)\|_{L^\infty} \leq \sqrt{3} \|w^0\|_{L^1} \text{vol}(B_{\mathbb{R}^{m-r_j}}(0, 1)) \int_{\xi' \in B_{\mathbb{R}^{r_j}}(0,d)} e^{-cc_j |\xi'|^{\alpha_j} t} d\xi'.$$

Performing the change of variable $\zeta = t^{1/\alpha_j} \xi'$, we get

$$\int_{\xi' \in B_{\mathbb{R}^{r_j}}(0,d)} e^{-cc_j |\xi'|^{\alpha_j} t} d\xi' \leq \frac{1}{t^{r_j/\alpha_j}} \int_{\mathbb{R}^{r_j}} e^{-cc_j |\zeta|^{\alpha_j}} d\zeta,$$

which gives (48) with any constant C such that

$$C \geq \sqrt{3} \sum_{j=1}^J \text{vol}(B_{\mathbb{R}^{m-r_j}}(0, 1)) \int_{\mathbb{R}^{r_j}} e^{-cc_j |\zeta|^{\alpha_j}} d\zeta.$$

□

Remark 12. Extending the validity of this kind of decomposition to more general situations (that is when the algebraic submanifold $\mathcal{D}(A_1, \dots, A_m, B)$ is not the union of a finite number of vector subspaces) is an open problem. Indeed, the strategy used in the proof would need a parameterization of the submanifold $\mathcal{D}(A_1, \dots, A_m, B)$. In this case the conditions to be imposed on the initial datum to ensure a similar decomposition are also to be clarified.

4.2. Decomposition when $n_1 = 1$

This subsection is devoted to studying the function $\omega \in S^{m-1} \mapsto N_{*,\epsilon}(\omega)$ for $\epsilon > 0$, defined in Proposition 1 in the particular case $n_1 = 1$ (that is $\text{Ker}(B) = \text{Span}(e_1)$). The parameters $\epsilon, m_0, \dots, m_{n-1}$ are fixed as in Proposition 1.

Proposition 7. *Let A_1, \dots, A_m, B be $n * n$ real matrices with A_1, \dots, A_m symmetric and B of the form (5) with $n_1 = 1$. Then*

- (1) $\mathcal{D}(A_1, \dots, A_m, B)$ is a vector subspace,
- (2) there exists $\epsilon_* = \epsilon_*(A_1, \dots, A_m, B) > 0$ such that, for every $\epsilon \in (0, \epsilon_*)$, there exists $c = c(\epsilon, A_1, \dots, A_m, B) > 0$ such that, for every $\omega \in S^{m-1}$,

$$N_{*,\epsilon}(\omega) \geq c \text{dist}(\omega, \mathcal{D})^2.$$

A direct consequence of Proposition 7 and Proposition 6 is the following theorem.

Theorem 2. *Under the same assumptions as in Proposition (7), there exists $C, \lambda > 0$ such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n)$ with (46), the solution of (4) with initial condition w^0 can be decomposed as*

$$w = w_1 + w_2 + w_3 + w_4$$

where (14) holds and

$$\|w_3(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{\sqrt{t}} N_1(w^0), \quad \forall t \in (0, +\infty) \tag{51}$$

$$\|w_4(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{t^{\frac{r}{2}}} \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}, \quad \forall t \in (0, +\infty), \tag{52}$$

where $r := \text{codim}(\mathcal{D}(A_1, \dots, A_m, B))$.

Proof of Proposition 7. The assumption $n_1 = 1$ implies that

$$\begin{aligned} \mathcal{D}(A_1, \dots, A_m, B) &= \{\xi \in \mathbb{R}^m; e_1 \text{ is an eigenvector of } A(\xi)\} \\ &= \{\xi \in \mathbb{R}^m; \sum_{j=1}^m a_{k,1}^{(j)} \xi_j = 0, \forall k = 2, \dots, n\}, \end{aligned} \tag{53}$$

which proves (1).

Let $\epsilon_* > 0$ be such that

$$\frac{2\mathcal{M}_1^2}{c} \epsilon_*^{m_1} < \frac{1}{2},$$

$$\epsilon_*^{m_1} \sum_{k=2}^n \left(\sum_{j=1}^m |a_{k,1}^{(j)}| \right)^2 < 1,$$

where \mathcal{M}_1 is defined by (27) and $c = c(B)$ is such that $|Bv|^2 \geq c|v|^2, \forall v \in \text{Span}(e_2, \dots, e_n)$. We have

$$N_*(\omega) = \min \left\{ \sum_{k=0}^{n-1} \epsilon^{mk} |BA(\omega)^k x|^2; x \in S^{n-1} \right\}$$

$$\geq \min \left\{ |Bx|^2 + \epsilon^{m_1} |BA(\omega)x|^2; x \in S^{n-1} \right\}.$$

Let $\omega \in \mathbb{R}^m - \mathcal{D}(A_1, \dots, A_m, B)$ and $x \in S^{n-1}$. We consider the decomposition $x = \beta e_1 + x_\perp$ where $\beta \in \mathbb{R}$ and $x_\perp \in \text{Span}(e_2, \dots, e_n)$. Thanks to the definition of c and (27), we have

$$|Bx|^2 + \epsilon^{m_1} |BA(\omega)x|^2 = |Bx_\perp|^2 + \beta^2 \epsilon^{m_1} |BA(\omega)e_1|^2$$

$$+ \epsilon^{m_1} |BA(\omega)x_\perp|^2 + 2\beta \epsilon^{m_1} \langle BA(\omega)e_1, BA(\omega)x_\perp \rangle$$

$$\geq c|x_\perp|^2 + \beta^2 \epsilon^{m_1} |BA(\omega)e_1|^2 - 2|\beta| \epsilon^{m_1} |BA(\omega)e_1| \mathcal{M}_1 |x_\perp|$$

$$\geq \frac{c}{2} |x_\perp|^2 + \beta^2 \epsilon^{m_1} \left(1 - \frac{2\mathcal{M}_1^2}{c} \epsilon^{m_1} \right) |BA(\omega)e_1|^2$$

$$\geq \frac{c}{2} |x_\perp|^2 + \frac{\beta^2}{2} \epsilon^{m_1} |BA(\omega)e_1|^2.$$

Since $Be_1 = 0$, we have

$$BA(\omega)e_1 = B \begin{pmatrix} 0 \\ \sum_{j=1}^m a_{2,1}^{(j)} \omega_j \\ \dots \\ \sum_{j=1}^m a_{n,1}^{(j)} \omega_j \end{pmatrix}$$

thus there exists a positive constant $c = c(B)$ such that

$$|BA(\omega)e_1|^2 \geq c \sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \omega_j \right|^2.$$

Therefore,

$$|Bx|^2 + \epsilon^{m_1} |BA(\omega)x|^2 \geq \frac{c}{2} \min \left\{ 1; \epsilon^{m_1} \sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \omega_j \right|^2 \right\} (|x_\perp|^2 + \beta^2),$$

thus

$$N_*(\omega) \geq \frac{\epsilon^{m_1 c}}{2} \sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \omega_j \right|^2.$$

In order to conclude, let us prove the existence of a constant $C > 0$ such that

$$\sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \omega_j \right|^2 \geq C \text{dist}(\omega, \mathcal{D})^2.$$

Let \mathcal{D}^\perp be the orthogonal supplementary of the subspace \mathcal{D} . Thanks to (53), the quadratic form

$$\tilde{\omega} \in \mathcal{D}^\perp \mapsto \sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \tilde{\omega}_j \right|^2$$

is positive definite on \mathcal{D}^\perp . Thus, there exists $C > 0$ such that

$$\sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \tilde{\omega}_j \right|^2 \geq C |\tilde{\omega}|^2, \quad \forall \tilde{\omega} \in \mathcal{D}^\perp.$$

Using the decomposition

$$\begin{aligned} \mathbb{R}^m &= \mathcal{D} \oplus \mathcal{D}^\perp \\ \omega &= \omega^\sharp + \tilde{\omega} \end{aligned}$$

we get

$$\sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \omega_j \right|^2 = \sum_{k=2}^n \left| \sum_{j=1}^m a_{k,1}^{(j)} \tilde{\omega}_j \right|^2 \geq C |\tilde{\omega}|^2 = C \text{dist}(\omega, \mathcal{D})^2, \quad \forall \omega \in S^{m-1}.$$

□

4.3. Discussion of explicit examples

This section is devoted to the study of explicit examples and to discussing the optimal value of the exponents α_j in the assumption (45) of Proposition 6. Note that the optimal values of the exponents α_j are important in the study of the asymptotic behavior of (4) because they appear in the decomposition of these solutions (see Proposition 6).

Consider the case $m \geq 2$,

$$A_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$A_k = 0$ for $k = 3, \dots, m$ and the associated system (4) with $\xi \in \mathbb{R}^m$. Then, we have $n_1 = 2$ and, for every $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{R}^m$,

$$A(\xi) = \sum_{j=1}^m \xi_j A_j = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{pmatrix}.$$

Thus

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$$

where

$$\mathcal{D}_1 := \{\xi \in \mathbb{R}^m; \xi_1 = 0\}, \quad \mathcal{D}_2 := \{\xi \in \mathbb{R}^m; \xi_2 = 0\}.$$

(see the analysis made in Example 2.a in Section 3.2).

Proposition 8. *There exists $\epsilon_* > 0$ such that, for every $\epsilon \in (0, \epsilon_*)$,*

(1) *with $m = 2$, there exists $c > 0$ such that,*

$$N_*(\omega) \geq c \text{dist}(\omega, \mathcal{D})^2, \quad \forall \omega \in S^1. \tag{54}$$

(2) *with $m \geq 3$, (54) fails but there exists $c > 0$ such that*

$$N_*(\omega) \geq c \text{dist}(\omega, \mathcal{D})^4, \quad \forall \omega \in S^{m-1}. \tag{55}$$

The previous proposition leads to the following remarks

- The assumption $n_1 = 1$ is not necessary for (54) to hold. Indeed, it holds for the example studied in this section with $m = 2$, and in that case $n_1 = 2$.
- The power α_j in (45), which coincides with 2 when $n_1 = 1$, may be strictly greater than 2 when $n_1 \geq 2$. Indeed, for the example studied in this section with $m \geq 3$, (45) holds with $\alpha_j = 4$ but not with $\alpha_j = 2$.

Propositions 8 and 6 lead to the following decomposition for the solutions of (4) with the previous matrices.

Theorem 3. *Let $m \geq 2$ and A_1, \dots, A_m, B be defined as at the beginning of this section. There exists $C, \lambda > 0$ such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n)$ with (46), the solution of (4) with initial condition w^0 can be decomposed as*

$$w = w_1 + w_2 + w_3 + w_4$$

where (14) holds and

- when $m = 2$,

$$\|w_3(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{\sqrt{t}} N_1(w^0), \quad \forall t \in (0, +\infty), \tag{56}$$

$$\|w_4(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{t^{\frac{m-1}{2}}} \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}, \quad \forall t \in (0, +\infty), \tag{57}$$

– when $m \geq 3$,

$$\|w_3(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{t^{\frac{1}{4}}} N_1(w^0), \quad \forall t \in (0, +\infty), \quad (58)$$

$$\|w_4(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^n)} \leq \frac{C}{t^{\frac{m-1}{4}}} \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)}, \quad \forall t \in (0, +\infty). \quad (59)$$

Proof of Proposition 8. Let $\epsilon_* > 0$ be such that

$$\epsilon_*^{m_2} < \frac{1}{2},$$

and $\epsilon \in (0, \epsilon_*)$. Easy computations lead to

$$M_\epsilon(\omega) = \sum_{k=0}^2 \epsilon^{m_k} A(\omega)^k B^t B A(\omega)^k = \begin{pmatrix} \epsilon^{m_1} \omega_2^2 & 0 & 0 \\ 0 & \epsilon^{m_2} \omega_1^2 \omega_2^2 & \epsilon^{m_2} \omega_1 \omega_2^3 \\ 0 & \epsilon^{m_2} \omega_1 \omega_2^3 & 1 + \epsilon^{m_2} \omega_2^4 \end{pmatrix} \quad (60)$$

Thus, for every $v \in S^2$, we have

$$\begin{aligned} v^t M(\omega) v &= \epsilon^{m_1} \omega_2^2 v_1^2 + \epsilon^{m_2} \omega_1^2 \omega_2^2 v_2^2 + 2\epsilon^{m_2} \omega_1 \omega_2^3 v_2 v_3 + [1 + \epsilon^{m_2} \omega_2^4] v_3^2 \\ &\geq \epsilon^{m_1} \omega_2^2 v_1^2 + \frac{1}{2} \epsilon^{m_2} \omega_1^2 \omega_2^2 v_2^2 + [1 - \epsilon^{m_2} \omega_2^4] v_3^2 \\ &\geq \min\{\epsilon^{m_1} \omega_2^2, \frac{1}{2} \epsilon^{m_2} \omega_1^2 \omega_2^2, 1 - \epsilon^{m_2} \omega_2^4\}. \end{aligned}$$

Noticing that, for every $\omega \in S^{m-1}$,

$$1 - \epsilon^{m_2} \omega_2^4 \geq 1 - \epsilon^{m_2} > \frac{1}{2} > \frac{1}{2} \epsilon^{m_2} \omega_1^2 \omega_2^2,$$

and using the definition (24), we get

$$N_*(\omega) \geq \min \left\{ \epsilon^{m_1} \omega_2^2, \frac{1}{2} \epsilon^{m_2} \omega_1^2 \omega_2^2 \right\}. \quad (61)$$

1st case: $m = 2$. For every $\omega \in S^1$, we have $\omega_1^2 + \omega_2^2 = 1$ thus (61) leads to

$$N_*(\omega) \geq \min\{\epsilon^{m_1}, \frac{1}{4} \epsilon^{m_2}\} \text{dist}(\omega, \mathcal{D})^2.$$

2nd case: $m \geq 3$. Let $\omega \in S^{m-1}$ and $j \in \{1, 2\}$ be such that $\text{dist}(\omega, \mathcal{D}) = \text{dist}(\omega, \mathcal{D}_j)$. Then $|\omega_j| \leq |\omega_i| \leq 1$ for $i \in \{1, 2\}$ such that $i \neq j$, thus (61) leads to

$$N_*(\omega) \geq \min \left\{ \epsilon^{m_1}, \frac{1}{2} \epsilon^{m_2} \right\} |\omega_j|^4 = \min \left\{ \epsilon^{m_1}, \frac{1}{2} \epsilon^{m_2} \right\} \text{dist}(\omega, \mathcal{D})^4.$$

Now, let us show that (54) may not hold with $m \geq 3$. We consider $\omega \in S^{m-1}$ of the form

$$\omega = \begin{pmatrix} \omega_1 \\ 2\omega_1 \\ \sqrt{1 - 5\omega_1^2} \\ 0 \\ \dots \\ 0 \end{pmatrix}, \tag{62}$$

where $\omega_1 > 0$ is small. Using the definition of $N_{*,\epsilon}(\omega)$ given by (24) and the explicit expression of the matrix $M_\epsilon(\omega)$ given by (60), we get

$$N_*(\omega) \leq e_2^t M_\epsilon(\omega) e_2 = 4\epsilon^{m_2} \omega_1^4 = 4\epsilon^{m_2} \text{dist}(\omega, \mathcal{D})^4.$$

We conclude by considering points $\omega \in S^{m-1}$ of the form (62), with $\omega_1 \rightarrow 0$. \square

4.4. Complete classification in the case $n = 2$

Let us consider the system (4) with

$$A_j := \begin{pmatrix} a_1^{(j)} & a_{1,2}^{(j)} \\ a_{1,2}^{(j)} & a_2^{(j)} \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{63}$$

with $a_1^{(j)}, a_{1,2}^{(j)}, a_{2,1}^{(j)}, a_2^{(j)} \in \mathbb{R}$. All possible partially dissipated systems with $n = 2$ enter in this setting. Obviously, in this case, $n_1 = n_2 = 1$.

Theorem 4. *Let $m \in \mathbb{N}^*$, $m \geq 2$, and A_1, \dots, A_m, B be real matrices of the form (63).*

- (A) *When $a_{1,2}^{(j)} = 0$, for $j = 1, \dots, m$, then, $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$, and, for every solution of (4) with $L^2(\mathbb{R}^m, \mathbb{R}^2)$ initial condition, the first component is a traveling wave and the second one decays exponentially in L^2 .*
- (B) *Let us assume that there exists $j \in \{1, \dots, m\}$ such that $a_{1,2}^{(j)} \neq 0$. Then $\mathcal{D}(A_1, \dots, A_m, B) = \{\xi \in \mathbb{R}^m; \sum_{j=0}^m a_{1,2}^{(j)} \xi_j = 0\}$ is an hyperplane of \mathbb{R}^m and there exist $C = C(A_1, \dots, A_m, B) > 0$ and $\lambda = \lambda(A_1, \dots, A_m, B) > 0$ such that, for every $w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^2)$ with (46), the solution of (4) with initial condition w^0 can be decomposed as*

$$w = w_1 + w_2 + w_3$$

where (14) holds and

$$\|w_3(t)\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^2)} \leq \frac{C}{\sqrt{t}} (N_1(w^0) + \|w^0\|_{L^1(\mathbb{R}^m, \mathbb{R}^n)})$$

Proof of Proposition 4. (A) can be obtained from easy explicit computations and (B) is a consequence of Theorem 2. \square

5. Summary array of the classification, open problems and conjectures

5.1. Summary array of the classification

We recapitulate the results proved in Sections 2, 3 and 4 in the following array.

m, n	(SK)	\mathcal{D}	L^2 stability	Decomposition
$m = 1$	Yes	$\{0\}$	Yes	$e^{-t} + \frac{1}{\sqrt{t}}$
$\forall n$	No	\mathbb{R}	No	$e^{-t} + \frac{1}{\sqrt{t}} + \text{tr. wave}$
$n = 2$	Yes	$\{0\}$	Yes	$e^{-t} + \frac{1}{t^{\frac{m}{2}}}$
$\forall m$	No	Hyperplane	Yes	$e^{-t} + \frac{1}{t^{\frac{m}{2}}} + \frac{1}{\sqrt{t}}$
	No	\mathbb{R}^m	No	$e^{-t} + \text{tr. wave}$
$\forall n$	Yes	$\{0\}$	Yes	$e^{-t} + \frac{1}{t^{\frac{m}{2}}}$
$\forall m$	No	\cup versus codim r	Yes	Conjecture: $e^{-t} + \frac{1}{t^{\frac{m}{2}}} + \frac{1}{t^{\frac{r}{2(n-1)}}} + \frac{1}{t^{\frac{1}{2(n-1)}}}$
	No	Submanifold	Yes	Open problem
	No	\mathbb{R}^m	No	Open problem
$\forall n$	Yes	$\{0\}$	Yes	$e^{-t} + \frac{1}{t^{\frac{m}{2}}}$
$\forall m$	No	Versus codim r	Yes	$e^{-t} + \frac{1}{t^{\frac{m}{2}}} + \frac{1}{t^{\frac{r}{2}}} + \frac{1}{\sqrt{t}}$
$n_1 = 1$				

In this array, by the expression “tr. waves” in the second line we mean “a sum of a finite number of traveling waves”, and in the fifth one “a traveling wave”. The notation “ \cup vs codim r ” in the seventh line means that $\mathcal{D}(A_1, \dots, A_m, B)$ is the union of a finite number of vector subspaces \mathcal{D}_j , $j = 1, \dots, J$ such that $\text{codim}(\mathcal{D}_j) \geq r$ for $j = 1, \dots, J$. The expression “vs codim r ” in the last line means that $\mathcal{D}(A_1, \dots, A_m, B)$ is a vector subspace with codimension r .

In the first and second lines we collect the results of Proposition 5. The third, fourth and fifth lines refer to Theorem 4. The sixth line comes from Theorem 1. The seventh line is explained in the next subsection. The eighth and ninth lines summarize the results of Propositions 2, 3 and 4. Finally, the tenth line contains the results of Theorem 2.

5.2. Open problems and conjectures

The decomposition of the solutions of (4) when $m \geq 2, n \geq 3$ are arbitrary is an open problem.

The simplest situation not covered in this article should be the case when the set of degeneracy is a finite union of vector subspaces of \mathbb{R}^m . We conjecture that, in that case, the following inequality holds

$$N_{*,\epsilon}(\omega) \geq c \text{dist}(\omega, \mathcal{D})^{2(n-1)}. \tag{64}$$

The intuition of the exponent $2(n - 1)$ comes from the definition

$$N_{*,\epsilon}(\omega) := \min \left\{ \sum_{k=0}^{n-1} \epsilon^{mk} |BA(\omega)^k x|^2; x \in S^{n-1} \right\}.$$

Indeed, for every $x \in S^{n-1}$, $\sum_{k=0}^{n-1} \epsilon^{mk} |BA(\omega)^k x|^2$ is polynomial of ω with degree $\leq 2(n - 1)$. If the conjecture (64) is valid, then Proposition 6 leads to a decomposition for the solutions of (4) when $\mathcal{D}(A_1, \dots, A_m, B)$ is a finite union of vector subspaces of \mathbb{R}^m . This decomposition is written in the seventh line of the array.

Similarly, when the set of degeneracy is a strict algebraic submanifold of \mathbb{R}^m of codimension r , we conjecture that any solution of (4) can be decomposed as $w = w_1 + w_2 + w_3 + w_4$ where w_1 and w_2 satisfy (14) and

$$\begin{aligned} \|w_3(t)\|_{L^\infty} &\leq \frac{C}{t^{\frac{1}{2(n-1)}}} N(w_0), \\ \|w_4(t)\|_{L^\infty} &\leq \frac{C}{t^{\frac{r}{2(n-1)}}} \|w\|_{L^1}, \end{aligned}$$

where the constant C depends on the parameterization of the algebraic manifold $\mathcal{D}(A_1, \dots, A_m, B)$. However, in this situation, it would not be sufficient to understand the behavior of $N_{*,\epsilon}(\omega)$, since one would also need the analogue of Proposition 6.

Another open problem is the nature of the nondissipated solutions and the decomposition of the solutions when $\mathcal{D}(A_1, \dots, A_m, B) = \mathbb{R}^m$.

6. Global existence around a constant equilibrium for the nonlinear system

6.1. Problem formulation

In this section, we study nonlinear systems of balance laws of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^m \frac{\partial F_j(w)}{\partial x_j} = Q(w) \tag{65}$$

where $m, n \in \mathbb{N}^*$, $w : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $w = w(t, x)$ is the unknown, $F_j, Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are smooth functions, and

$$Q(w) := \begin{pmatrix} 0 \\ q(w) \end{pmatrix}$$

where $0 \in \mathbb{R}^{n_1}$ and $q(w) \in \mathbb{R}^{n_2}$.

We consider a constant equilibrium $W_e \in \mathbb{R}^n$, that is, $Q(W_e) = 0$. The aim of this section is to investigate under what conditions the source term may prevent the breakdown of smooth solutions in a neighborhood of W_e . This question has been addressed earlier, for instance, in [36], under the following assumptions **(H0)**–**(H2)** in order to deal with strictly entropy dissipative symmetrizable systems.

- (H0):** The differential $d_{w_2}q(W_e)$ is invertible.
- (H1):** There exists a strictly convex entropy $\eta = \eta(w)$, defined in a convex compact neighborhood G of W_e , such that $d_w^2\eta(w)d_w F_j(w)$ is symmetric for every $w \in G$ and for every $j \in \{1, \dots, m\}$.

(H2): There exists a constant $C_G > 0$ such that, for every $w \in G$,

$$[d_w \eta(w) - d_w \eta(W_e)] \cdot Q(w) \leq -C_G |Q(w)|^2, \quad \forall w \in G. \quad (66)$$

Under these conditions, in [36] the existence of global smooth solutions in a neighborhood of W_e was proved, when the linearized system around W_e satisfies (SK).

We shall also assume that hypotheses **(H0)**–**(H2)** are fulfilled. Moreover, in order to simplify the proof, we will assume the following assumption **(H3)**.

(H3): There exists $D \in \mathbb{R}^{n_2 \times n_2}$ positive definite such that

$$Q(w) := \begin{pmatrix} 0 \\ -Dw_2 \end{pmatrix}. \quad (67)$$

In Section 6.2, we make precise Yong’s statement in [36] by giving an explicit estimate of the size of the neighborhood of W_e in which global existence holds. The possibility of measuring this size explicitly is a key ingredient in the degenerate case, studied in Section 6.3, where we show that the (SK) condition on the linearized system around W_e is not necessary for global existence around W_e . More precisely, in Section 6.3, we consider a constant degenerate equilibrium W_e and we assume the existence of a sequence of nondegenerate equilibria $(W_e^p)_{p \in \mathbb{N}}$ converging to W_e and such that the quantity N_{*, W_e^p} in (24) measuring the decay rate (25) for the linearized system around W_e^p converges to zero slowly enough. Then, we prove the existence of global smooth solutions for (65) in a neighborhood of W_e . Our theorem takes advantage of the contribution of the nonlinearity, when the linearized system is degenerate. Our proof is inspired by Coron’s return method for nonlinear control (see [7]). We end this section with an example of application of this theorem.

6.2. Size of the neighborhood for global existence under (SK)

In this section, we assume that (SK) holds. More precisely, it is supposed that the following hypothesis is fulfilled.

(H4): The linearized system around W_e satisfies (SK),

$$\text{Ker}(B) \cap \{\text{eigenvectors of } A_{W_e}(\omega)\} = \{0\}, \quad \forall \omega \in S^{m-1},$$

where $A_{W_e}(\omega) := \sum_{j=1}^m \omega_j A_j(W_e)$, $A_j(W_e) := d_w F_j(W_e)$ and

$$B := \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Thus, using the same notation as in Proposition 1, we deduce that

$$N_{*, W_e} := \min \left\{ \sum_{k=0}^{n-1} \epsilon^{mk} |B A_{W_e}(\omega)^k x|^2; x \in S^{n-1}, \omega \in S^{m-1} \right\} > 0.$$

We now introduce the compensating function

$$K_{W_e}(\omega) := \sum_{k=1}^{n-1} \epsilon^{m_k} [A_{W_e}(\omega)^{*k} B^* B A_{W_e}(\omega)^{k-1} - A_{W_e}(\omega)^{*k-1} B^* B A_{W_e}(\omega)^k]$$

with the notations of Proposition 1 (see the definition of a compensation function in Remark 5). We know that

$$\frac{B^* + B}{2} + \frac{1}{2} (K_{W_e}(\omega) A_{W_e}(\omega) + (K_{W_e}(\omega) A_{W_e}(\omega))^*) \geq \frac{1}{2} N_{*, W_e}. \tag{68}$$

We introduce

$$C_{W_e} := \max\{\|K_{W_e}(\omega)\|; \omega \in S^{m-1}\}, \tag{69}$$

$$\delta_{W_e} := \frac{1}{2} \min \left\{ \frac{1}{C_{W_e}}, \frac{N_{*, W_e}}{2C_{W_e}^2 \|D\|^2 + \|D\| N_{*, W_e}} \right\}. \tag{70}$$

Theorem 5. *We assume that (H0), (H1), (H2), (H3) and (H4) are fulfilled. Let $s \geq [m/2] + 2$ be an integer and $\mathcal{M} > 0$ be such that*

$$\|Q\|_{C^s(G)} + \sum_{j=1}^m \|A_j\|_{C^s(G)} \leq \mathcal{M}. \tag{71}$$

Then, there exist $\nu_p = \nu_p(\eta, G, \mathcal{M}) > 0$ for $p = 1, 2$ such that, for every $w_0 \in W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)$ with

$$\|w_0 - W_e\|_{H^s(\mathbb{R}^m, \mathbb{R}^n)} < \nu_1 \min \left\{ \delta_{W_e} N_{*, W_e}, \frac{N_{*, W_e}}{C_{W_e}}, 1 \right\},$$

the system (65) has a unique global solution

$$w \in C^0([0, +\infty), W_e + H^s(\mathbb{R}^m, \mathbb{R}^n))$$

satisfying, for every $T > 0$,

$$\|w(T) - W_e\|_{H^s}^2 + \int_0^T \|w_2\|_{H^s}^2 + \delta_{W_e} N_{*, W_e} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 \leq \nu_2 \|w_0 - W_e\|_{H^s}^2. \tag{72}$$

We refer to [13] for examples of application of this result in one dimension ($m = 1$).

Proof of Theorem 5. Thanks to the assumption (H3), we have $W_{e2} = 0$. We proceed in several steps.

Step 1: Let $w_0 \in W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)$. From the local in time existence theory of H^s solutions, we know that there exists $T > 0$ depending only on $\|w_0 - W_e\|_{H^s}$ such that (65), with initial value w_0 , has a unique solution

$$w \in C^0([0, T], W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)).$$

As long as (72) holds, one has

$$\|w(T) - W_e\|_{H^s(\mathbb{R}^m, \mathbb{R}^n)} \leq \sqrt{v_2} \|w_0 - W_e\|_{H^s}.$$

Since the right-hand side of this estimate does not depend on T , the solution can be continued up to $T = +\infty$. Thus, to prove the theorem, we only need to justify the bound (72).

Step 2: Let us now prove (72). Arguing as in [36, pp 255-259] we get the existence of $C_p = C_p(\eta, G, \mathcal{M}) > 0$, for $p = 1, 2$ such that

$$\begin{aligned} & \|w(T) - W_e\|_{H^s}^2 + \int_0^T \|w_2(t)\|_{H^s}^2 dt \\ & \leq C_1 \|w_0 - W_e\|_{H^s}^2 + C_2 N_s(T) \int_0^T \left(\|\nabla_x w(t)\|_{H^{s-1}}^2 + \|w_2(t)\|_{H^{s-1}}^2 \right) dt \end{aligned} \quad (73)$$

where

$$N_s(T) := \sup_{0 \leq t \leq T} \|w(t) - W_e\|_{H^s}.$$

In order to control the term $\int_0^T \|\nabla_x w(t)\|_{H^{s-1}}^2 dt$, we use the linearized system around W_e . Here we develop the arguments in [36] making them more precise. We have

$$\frac{\partial w}{\partial t} + \sum_{j=1}^m A_j(W_e) \frac{\partial w}{\partial x_j} = Q(w) + h_{W_e}$$

where

$$h_{W_e} := \sum_{j=1}^m [A_j(W_e) - A_j(w)] \frac{\partial w}{\partial x_j}.$$

Applying the Fourier transform, we get

$$\frac{\partial \hat{w}}{\partial t} + i A_{W_e}(\xi) \hat{w} = \widehat{Q(w)} + \hat{h}_{W_e}. \quad (74)$$

We compute the imaginary part of the scalar product in \mathbb{C}^n of Equation (74) with $K_{W_e}(\omega) \hat{w}(t, \xi)$ (where $\xi = |\xi| \omega$). Using the skew symmetry of $K_{W_e}(\omega)$, we get

$$\Im \left(\left\langle K_{W_e}(\omega) \hat{w}, \frac{\partial \hat{w}}{\partial t} \right\rangle \right) + |\xi| \langle \hat{w}, K_{W_e}(\omega) A_{W_e}(\omega) \hat{w} \rangle = \Im \langle K_{W_e}(\omega) \hat{w}, \widehat{Q(w)} + \hat{h}_{W_e} \rangle.$$

Thanks to (68), (69) and Cauchy–Schwarz inequality, we get

$$\begin{aligned} |\xi| \left(\frac{N_{s, W_e}}{2} |\hat{w}|^2 - \|D\| |\hat{w}_2|^2 \right) & \leq -\Im \langle K_{W_e}(\omega) \hat{w}, \frac{\partial \hat{w}}{\partial t} \rangle + C_{W_e} |\hat{w}| (|\widehat{Q(w)}| + |\hat{h}_{W_e}|) \\ & \leq -\Im \langle K_{W_e}(\omega) \hat{w}, \frac{\partial \hat{w}}{\partial t} \rangle + \frac{N_{s, W_e} |\xi|}{4} |\hat{w}|^2 \\ & \quad + \frac{2C_{W_e}^2}{N_{s, W_e} |\xi|} (|\widehat{Q(w)}|^2 + |\hat{h}_{W_e}|^2). \end{aligned}$$

Multiplying this inequality by $|\xi|^{2k-1}$, integrating over $(0, T)_t \times \mathbb{R}_\xi^m$, and summing over $k \in \{1, \dots, s\}$, we get

$$\begin{aligned} \frac{N_{*,W_e}}{4} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt &\leq C_{W_e} [\|w(T) - W_e\|_{H^s}^2 + \|w_0 - W_e\|_{H^s}^2] \\ &\quad + \int_0^T \|D\| \|w_2\|_{H^s}^2 dt + \frac{2C_{W_e}^2}{N_{*,W_e}} \\ &\quad \times \int_0^T \|Q(w)\|_{H^{s-1}}^2 + \|h_{W_e}\|_{H^{s-1}}^2 dt. \end{aligned} \quad (75)$$

Indeed, we have

$$\begin{aligned} &\sum_{k=1}^s \int_0^T \int_{\mathbb{R}^m} |\xi|^{2k-1} \Im \langle K_{W_e}(\omega) \hat{w}, \frac{\partial \hat{w}}{\partial t} \rangle d\xi dt \\ &= \sum_{k=1}^s \int_{\mathbb{R}^m} |\xi|^{2k-1} (\langle K_{W_e}(\omega) \hat{w}(T), \hat{w}(T) \rangle - \langle K_{W_e}(\omega) \hat{w}(0), \hat{w}(0) \rangle) d\xi \\ &\leq C_{W_e} \sum_{k=1}^s \int_{\mathbb{R}^m} |\xi|^{2k-1} (|\hat{w}(T)|^2 + |\hat{w}_0|^2) d\xi \\ &\leq C_{W_e} (\|w(T) - W_e\|_{H^s}^2 + \|w_0 - W_e\|_{H^s}^2). \end{aligned}$$

Let $\tilde{\mathcal{C}}_{\mathcal{M}} > 0$ be such that, for every $t \in (0, T)$,

$$\|h_{W_e}(t)\|_{H^{s-1}} \leq \sum_{j=1}^m |A_j|_{C^{s-1}(G)} N_s(T) \left\| \frac{\partial w}{\partial x_j} \right\|_{H^{s-1}} \leq \tilde{\mathcal{C}}_{\mathcal{M}} N_s(T) \|\nabla_x w(t)\|_{H^{s-1}}.$$

We get

$$\begin{aligned} \frac{N_{*,W_e}}{4} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt &\leq C_{W_e} [\|w(T) - W_e\|_{H^s}^2 + \|w_0 - W_e\|_{H^s}^2] \\ &\quad + \left(\|D\| + \frac{2C_{W_e}^2 \|D\|^2}{N_{*,\epsilon}} \right) \int_0^T \|w_2\|_{H^s}^2 dt \\ &\quad + \frac{2C_{W_e}^2 \tilde{\mathcal{C}}_{\mathcal{M}}^2}{N_{*,W_e}} N_s(T)^2 \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt. \end{aligned} \quad (76)$$

Thus multiplying (76) by δ_{W_e} and adding (73), we get

$$\begin{aligned} &\frac{1}{2} \|w(T) - W_e\|_{H^s}^2 + \left(\frac{1}{2} - \mathcal{C}_2 N_s(T) \right) \int_0^T \|w_2\|_{H^s}^2 + \frac{\delta_{W_e} N_{*,W_e}}{4} \\ &\quad \times \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 \leq \left(\mathcal{C}_1 + \frac{1}{2} \right) \|w_0 - W_e\|_{H^s}^2 \\ &\quad + \left(\mathcal{C}_2 N_s(T) + 2\delta_{W_e} \tilde{\mathcal{C}}_{\mathcal{M}}^2 N_s(T)^2 \frac{C_{W_e}^2}{N_{*,W_e}} \right) \int_0^T \|\nabla_x w\|_{H^{s-1}}^2. \end{aligned} \quad (77)$$

Indeed, the definition of δ_{W_e} (see (70)) implies

$$\delta_{W_e} \left(\|D\| + \frac{2C_{W_e}^2 \|D\|^2}{N_{*,W_e}} \right) \leq \frac{1}{2}$$

and

$$\delta_{W_e} C_{W_e} \leq \frac{1}{2}. \tag{78}$$

So, as long as $N_s(T)$ is small enough so that $w(t, \cdot) \in G$ on $(0, T)$,

$$C_2 N_s(T) < 1/4, \tag{79}$$

$$C_2 N_s(T) + 2\delta_{W_e} \tilde{C}_{\mathcal{M}}^2 N_s(T)^2 \frac{C_{W_e}^2}{N_{*,W_e}} \leq \frac{\delta_{W_e} N_{*,W_e}}{8}, \tag{80}$$

we have (72) with $\nu_2 := 8(C_1 + 1)$ and

$$\|w(T) - W_e\|_{H^s} \leq \sqrt{8(C_1 + 1)} \|w_0 - W_e\|_{H^s}. \tag{81}$$

The inequality (80) holds in particular when

$$C_2 N_s(T) \leq \frac{\delta_{W_e} N_{*,W_e}}{16} \quad \text{and} \quad 2\delta_{W_e} \tilde{C}_{\mathcal{M}}^2 N_s(T)^2 \frac{C_{W_e}^2}{N_{*,W_e}} \leq \frac{\delta_{W_e} N_{*,W_e}}{16},$$

that is when

$$N_s(T) \leq N_{*,W_e} \min \left\{ \frac{\delta_{W_e}}{16C_2}, \frac{1}{4\sqrt{2}\tilde{C}_{\mathcal{M}}C_{W_e}} \right\}. \tag{82}$$

The inequality (81) justifies that (82) and (79) hold as soon as

$$\begin{aligned} \|w_0 - W_e\|_{H^s} &\leq \frac{1}{\sqrt{8(C_1 + 1)}} \frac{1}{4C_2}, \\ \|w_0 - W_e\|_{H^s} &< \frac{1}{\sqrt{8(C_1 + 1)}} N_{*,W_e} \min \left\{ \frac{\delta_{W_e}}{16C_2}, \frac{1}{4\sqrt{2}\tilde{C}_{\mathcal{M}}C_{W_e}} \right\}. \end{aligned}$$

We have proved Theorem 5 with

$$\nu_1 := \frac{1}{\sqrt{8(C_1 + 1)}} \min \left\{ \frac{1}{16C_2}, \frac{1}{4\sqrt{2}\tilde{C}_{\mathcal{M}}} \right\}, \quad \nu_2 := 8(C_1 + 1).$$

□

Remark 13. Thanks to the careful study of the linear hyperbolic systems in the previous sections, one sees that Yong’s strategy [36], as we have pursued it here, cannot be applied directly to the case where the linearized system around the equilibrium W_e does not satisfy (SK). For example, if the linearized system around W_e is such that

$$N_{*,W_e}(\omega) \geq c \text{dist}(\omega, \mathcal{D})^2,$$

we cannot get a control on the whole term $\int_0^T \|\nabla_x w(t)\|_{H^{s-1}}^2 dt$ but only on part of the gradient. This corresponds to the situation where some of the space directions are missing in the first order derivatives in the linearized system. In that case, the PDE does not have any structure on those missing variables which are only unstructured parameters, the gain of time-space estimates being impossible on those directions.

6.3. First application: global existence without (SK)

In this subsection, we assume the following hypothesis.

(H5) The linearized system around W_e does not satisfy (SK), but there exists a constant equilibrium $W_e^p \in G$ such that the linearized system around W_e^p satisfies (SK).

Theorem 6. *We assume that (H0)–(H3) and (H5) are fulfilled. Let $s \geq [m/2]+2$ be an integer and $\mathcal{M} > 0$ be such that (71) holds. There exist $\nu_k = \nu_k(\eta, G, \mathcal{M}) > 0$ for $k = 0, 1, 2$ such that, if*

$$N_*(W_e^p) > \nu_0 C_{W_e^p} |W_e^p - W_e|, \tag{83}$$

then, for every $w_0 \in W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)$ with

$$\|w_0 - W_e\|_{H^s(\mathbb{R}^m, \mathbb{R}^n)} < \nu_1 \min \left\{ \delta_{W_e^p} N_{*, W_e^p}, \frac{N_{*, W_e^p}}{C_{W_e^p}}, 1 \right\}, \tag{84}$$

the system (65) has a unique global solution $w \in C^0([0, +\infty), W_e + H^s(\mathbb{R}^m, \mathbb{R}^n))$ satisfying, for every $T > 0$,

$$\|w(T) - W_e\|_{H^s}^2 + \int_0^T \|w_2\|_{H^s}^2 + \delta_{W_e^p} N_{*, W_e^p} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 \leq \nu_2 \|w_0 - W_e\|_{H^s}^2. \tag{85}$$

Remark 14. Obviously (H5) implies that $N_{*, W_e^p} \rightarrow 0$ when $W_e^p \rightarrow W_e$. The assumption (83) only says that this convergence is not too fast.

Proof of Theorem 6. This proof is an adaptation of the proof of Theorem 5 and the same notation will be used.

As in the proof of Theorem 5, there exists $\mathcal{C}_p = \mathcal{C}_p(\eta, G, \mathcal{M}) > 0$, for $p = 1, 2$ such that (73) holds.

In order to control $\int_0^T \|\nabla_x w(t)\|_{H^{s-1}}^2 dt$, we use the linearized system around W_e^p , writing

$$\frac{\partial w}{\partial t} + \sum_{j=1}^m A_j(W_e^p) \frac{\partial w}{\partial x_j} = Q(w) + h_{W_e^p}.$$

Applying the Fourier transform, we get

$$\frac{\partial \hat{w}}{\partial t} + i A_{W_e^p}(\xi) \hat{w} = \widehat{Q(w)} + \hat{h}_{W_e^p}. \tag{86}$$

We compute the imaginary part of the scalar product of this equation with $K_{W_e^p}(\omega) \hat{w}(t, \xi)$. Performing the same analysis as in the proof of Theorem 5, we get

$$\begin{aligned}
\frac{N_{*,W_e^p}}{4} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt &\leq C_{W_e^p} [\|w(T) - W_e\|_{H^s}^2 + \|w_0 - W_e\|_{H^s}^2] \\
&\quad + \left(\|D\| + \frac{2C_{W_e^p}^2 \|D\|^2}{N_{*,W_e^p}} \right) \int_0^T \|w_2(t)\|_{H^s}^2 dt \\
&\quad + \frac{2C_{W_e^p}^2}{N_{*,W_e^p}} \int_0^T \|h_{W_e^p}(t)\|_{H^{s-1}}^2 dt. \tag{87}
\end{aligned}$$

Moreover, there exists $\tilde{C}_{\mathcal{M}} > 0$ such that for every $t \in (0, T)$,

$$\begin{aligned}
\|h_{W_e^p}(t)\|_{H^{s-1}} &= \|h_{W_e}(t) + \sum_{j=1}^m [A_j(W_e) - A_j(W_e^p)] \frac{\partial w}{\partial x_j}(t)\|_{H^{s-1}} \\
&\leq \tilde{C}_{\mathcal{M}} [N_s(T) + |W_e - W_e^p|] \|\nabla_x w(t)\|_{H^{s-1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{N_{*,W_e^p}}{4} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt &\leq C_{W_e^p} [\|w(T) - W_e\|_{H^s}^2 + \|w_0 - W_e\|_{H^s}^2] \\
&\quad + \left(\|D\| + \frac{2C_{W_e^p}^2 \|D\|^2}{N_{*,W_e^p}} \right) \int_0^T \|w_2(t)\|_{H^s}^2 dt \\
&\quad + \frac{4C_{W_e^p}^2 \tilde{C}_{\mathcal{M}}^2}{N_{*,W_e^p}} (N_s(T)^2 + |W_e - W_e^p|^2) \\
&\quad \times \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt. \tag{88}
\end{aligned}$$

Multiplying (88) by $\delta_{W_e^p}$ and adding (73), as long as $C_2 N_s(T) < 1/4$, we get

$$\begin{aligned}
&\frac{1}{2} \|w(T) - W_e\|_{H^s}^2 + \frac{1}{4} \int_0^T \|w_2\|_{H^s}^2 dt + \frac{\delta_{W_e^p} N_{*,W_e^p}}{4} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt \\
&\leq \left(C_1 + \frac{1}{2} \right) \|w_0 - W_e\|_{H^s}^2 \\
&\quad + \left(C_2 N_s(T) + 4\delta_{W_e^p} \tilde{C}_{\mathcal{M}}^2 [N_s(T)^2 + |W_e - W_e^p|^2] \frac{C_{W_e^p}^2}{N_{*,W_e^p}} \right) \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt.
\end{aligned}$$

So, as long as

$$C_2 N_s(T) + 4\delta_{W_e^p} \tilde{C}_{\mathcal{M}}^2 [N_s(T)^2 + |W_e - W_e^p|^2] \frac{C_{W_e^p}^2}{N_{*,W_e^p}} \leq \frac{\delta_{W_e^p} N_{*,W_e^p}}{8}, \tag{89}$$

we have (85) with $\nu_2 := 8(\mathcal{C}_1 + 1)$. The inequality (89) holds in particular when

$$\begin{aligned} \mathcal{C}_2 N_s(T) &\leq \frac{\delta_{W_e^p} N_{*, W_e^p}}{24}, \\ 4\delta_{W_e^p} \tilde{\mathcal{C}}_{\mathcal{M}}^2 \frac{C_{W_e^p}^2}{N_{*, W_e^p}} N_s(T)^2 &\leq \frac{\delta_{W_e^p} N_{*, W_e^p}}{24}, \\ 4\delta_{W_e^p} \tilde{\mathcal{C}}_{\mathcal{M}}^2 \frac{C_{W_e^p}^2}{N_{*, W_e^p}} |W_e - W_e^p|^2 &\leq \frac{\delta_{W_e^p} N_{*, W_e^p}}{24}, \end{aligned}$$

that is when

$$N_s(T) \leq \min \left\{ \frac{\delta_{W_e^p} N_{*, W_e^p}}{24\mathcal{C}_2}, \frac{N_{*, W_e^p}}{\sqrt{96}\mathcal{C}_{W_e^p} \tilde{\mathcal{C}}_{\mathcal{M}}} \right\}, \tag{90}$$

and

$$|W_e - W_e^p| \leq \frac{N_{*, W_e^p}}{\sqrt{96}\mathcal{C}_{W_e^p} \tilde{\mathcal{C}}_{\mathcal{M}}}.$$

The inequality (85) proves that (90) holds when

$$\|w_0 - W_e\|_{H^s} \leq \frac{1}{\sqrt{8(\mathcal{C}_1 + 1)}} \min \left\{ \frac{\delta_{W_e^p} N_{*, W_e^p}}{24\mathcal{C}_2}, \frac{N_{*, W_e^p}}{\sqrt{96}\mathcal{C}_{W_e^p} \tilde{\mathcal{C}}_{\mathcal{M}}} \right\}.$$

We have proved Theorem 6 with

$$\nu_0 := \sqrt{96}\tilde{\mathcal{C}}_{\mathcal{M}}, \quad \nu_1 := \frac{1}{\sqrt{8(\mathcal{C}_1 + 1)}} \min \left\{ \frac{1}{24\mathcal{C}_2}, \frac{1}{\sqrt{96}\tilde{\mathcal{C}}_{\mathcal{M}}} \right\}, \quad \nu_2 := 8(\mathcal{C}_1 + 1).$$

□

Of course, the assumption (83) may be difficult to check on a given system of conservation laws. But it gives an easy way to build examples of nonlinear systems of conservation laws with global smooth solutions in a neighborhood of a degenerate constant equilibrium.

Example of application of Theorem 6: Let us consider (65) with $m = 1, n = 2$,

$$F(w) = \begin{pmatrix} F^{(1)}(w) \\ F^{(2)}(w) \end{pmatrix}, \quad Q(w) = \begin{pmatrix} 0 \\ w_2 \end{pmatrix}, \quad W_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad W_e^p = \begin{pmatrix} a_p \\ 0 \end{pmatrix},$$

where $a_p \neq 0$ and $a_p \rightarrow 0$. We assume

–

$$\frac{\partial F^{(2)}}{\partial w_1}(W_e) = 0,$$

–

$$\frac{\partial F^{(1)}}{\partial w_2}(w) = \frac{\partial F^{(2)}}{\partial w_1}(w), \quad \forall w \in G, \tag{91}$$

where G is a neighborhood of W_e in \mathbb{R}^2 ,
 – there exists $M > 0$ such that

$$\left| \frac{\partial F^{(2)}}{\partial w_2}(W_e^p) \right| \leq M \left| \frac{\partial F^{(1)}}{\partial w_2}(W_e^p) \right|, \quad \forall p \in \mathbb{N}. \tag{92}$$

Assumption **(H0)** is fulfilled because $q(w) = -w_2$, thus $d_{w_2}q(w) = -1$ is invertible. assumption **(H1)** holds with $\eta(w) := |w|^2$ because $d_w F(w)$ is symmetric, thanks to assumption (91). assumption **(H2)** is fulfilled with $C_G = 1$. It is clear that **(H3)** holds.

By definition, $N_{*,W}$ is the smallest eigenvalue of the nonnegative matrix

$$\begin{pmatrix} \epsilon^{m_1} \left(\frac{\partial F^{(2)}}{\partial w_1}(W) \right)^2 & \epsilon^{m_1} \frac{\partial F^{(2)}}{\partial w_1}(W) \frac{\partial F^{(2)}}{\partial w_2}(W) \\ \epsilon^{m_1} \frac{\partial F^{(2)}}{\partial w_1}(W) \frac{\partial F^{(2)}}{\partial w_2}(W) & 1 + \epsilon^{m_1} \left(\frac{\partial F^{(2)}}{\partial w_2}(W) \right)^2 \end{pmatrix},$$

thus $N_{*,W_e} = 0$, that is (SK) is not satisfied for the linearized system around W_e and, for $\epsilon > 0$ small enough, we have

$$N_{*,W_e^p} \geq \frac{1}{2} \epsilon^{m_1} \left| \frac{\partial F^{(2)}}{\partial w_1}(W_e^p) \right|^2.$$

By definition of $C_{W_e^p}$ and thanks to (92), we have

$$C_{W_e^p} \leq 2\epsilon^{m_1} (1 + M) \left| \frac{\partial F^{(2)}}{\partial w_1}(W_e^p) \right|$$

Thus, (83) holds, in particular, when

$$\left| \frac{\partial F^{(1)}}{\partial w_2}(W_e^p) \right| > 4v_0(1 + M) |W_e^p - W_e|,$$

which is valid, in particular, when

$$\left| \frac{\partial^2 F^{(1)}}{\partial w_2 \partial w_1}(W_e) \right|$$

is large enough. At this point, it is clear that assumption (83) takes advantage of the nonlinearity.

6.4. Second application: uniform well-posedness

Let us consider a partially dissipative nonlinear hyperbolic system of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^m A_j(w_2) \frac{\partial w}{\partial x_j} = -\frac{1}{\tau} Bw, \tag{93}$$

where B has form (5), $\tau > 0$ is a relaxation parameter, $w = (w_1, w_2)$ with $w_1 \in \mathbb{R}^{n_1}$ and $w_2 \in \mathbb{R}^{n_2}$ and the matrices $A_j(w_2)$ are symmetric. To simplify, we assume that

$$\Re \langle DX, X \rangle \geq |X|^2, \quad \forall X \in \mathbb{C}^{n_2}.$$

Assume we are interested in the limit of the solutions w_τ of this system when the parameter τ tends to zero. Then, one needs a uniform well-posedness result with respect to the parameter τ that is, there exists $\delta > 0$ such that, for every $w_0 \in H^s$ with $\|w_0\|_{H^s} < \delta$ and for every $\tau \in (0, 1)$, the system (93) has a unique global solution. Such a situation arises in article [8] where the particular case of the multidimensional isothermal Euler equation is studied.

The goal of this section is to show that this uniform well-posedness can be proved thanks to the explicit size of the neighborhood for global existence proved in Section 6.2. Precisely, we will prove the following result.

Theorem 7. *We assume (H0), (H1), (H2), (H3) and (H4) with $W_e = 0$. Let $s \geq [m/2] + 2$ be an integer. Then, there exists $\nu_p = \nu_p(\eta, G, A_1, \dots, A_m, B) > 0$ for $p = 1, 2$ such that, for every $w_0 \in H^s(\mathbb{R}^m, \mathbb{R}^n)$ with $\|w_0\|_{H^s} \leq \nu_1$ and for every $\tau \in (0, 1)$, the system (93) has a unique global solution $w \in C^0([0, +\infty), H^s(\mathbb{R}^m, \mathbb{R}^n))$ satisfying, for every $T > 0$,*

$$\|w(T)\|_{H^s}^2 + \frac{1}{\tau} \int_0^T \|w_2(t)\|_{H^s}^2 dt + \tau \int_0^T \|\nabla w\|_{H^{s-1}}^2 dt \leq \nu_2 \|w_0\|_{H^s}^2. \tag{94}$$

The proof of Theorem 7 follows the one of Theorem 5; we only specify the behavior of the different constants with respect to the parameter τ . It also follows the proof of [8].

Proof of Theorem 7. In a first step, we enter Yong’s proof in order to specify the behavior of the constant appearing in (73) with respect to the parameter τ .

Using the entropy variable, as in [36], we get $C_1 = C_1(\eta, G, A_1, \dots, A_m, B) > 0$ such that

$$\|w(T)\|_{L^2}^2 + \frac{1}{\tau} \int_0^T \|w_2(t)\|_{L^2}^2 dt \leq C_1 \|w_0\|_{H^s}^2. \tag{95}$$

Let $\alpha \in \mathbb{N}^m$ such that $|\alpha| \leq s$. Applying ∂_x^α to Equation (93), and taking the scalar product in \mathbb{R}^m with $\partial_x^\alpha w$, we get

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \left(|\partial_x^\alpha w|^2 \right) + \sum_{j=1}^m \frac{1}{2} \frac{\partial}{\partial x_j} \left(\langle A_j(w_2) \partial_x^\alpha w, \partial_x^\alpha w \rangle \right) = -\frac{1}{\tau} \langle B \partial_x^\alpha w, \partial_x^\alpha w \rangle \\ & + \sum_{j=1}^m \left(\langle [A_j(w_2), \partial_x^\alpha] \partial_{x_j} w, \partial_x^\alpha w \rangle + \frac{1}{2} \langle \partial_{x_j} [A_j(w_2)] \partial_x^\alpha w, \partial_x^\alpha w \rangle \right). \end{aligned}$$

Integrating this equality over $(0, T) \times \mathbb{R}^m$ leads to

$$\frac{1}{2} \|\partial_x^\alpha w(T)\|_{L^2}^2 + \frac{1}{\tau} \int_0^T \|\partial_x^\alpha w_2(t)\|_{L^2}^2 dt \leq \frac{1}{2} \|\partial_x^\alpha w_0\|_{L^2}^2 + I_1 + I_2, \quad (96)$$

where

$$I_1 := \frac{1}{2} \int_0^T \int_{\mathbb{R}^m} \sum_{j=1}^m \langle \partial_{x_j} [A_j(w_2)] \partial_x^\alpha w, \partial_x^\alpha w \rangle dx dt,$$

$$I_2 := \int_0^T \int_{\mathbb{R}^m} \sum_{j=1}^m \langle [A_j(w_2), \partial_x^\alpha] \partial_{x_j} w, \partial_x^\alpha w \rangle dx dt.$$

There exists a constant $C_A > 0$ depending only on A_1, \dots, A_m such that

$$|I_1| \leq C_A \int_0^T \int_{\mathbb{R}^m} |\nabla w_2| |\partial_x^\alpha w|^2 dx dt$$

$$\leq C_A \int_0^T \|\nabla w_2\|_{L^2} \|\partial_x^\alpha w\|_{L^\infty} \|\partial_x^\alpha w\|_{L^2} dt.$$

Using the Sobolev inclusion $H^{s-1} \subset L^\infty$, we get

$$|I_1| \leq C_A N_s(T) \int_0^T \|\nabla w_2\|_{L^2} \|\partial_x^\alpha w\|_{L^2} dt$$

$$\leq \frac{C_A N_s(T)}{\tau} \int_0^T \|w_2(t)\|_{H^s}^2 dt + \tau C_A N_s(T) \int_0^T \|\nabla w(t)\|_{H^{s-1}}^2 dt. \quad (97)$$

Using [36, Lemma 3.2 page 255] and the same Sobolev inclusion, we get a constant $C'_A > 0$ depending only on A_1, \dots, A_m such that

$$|I_2| \leq \int_0^T \sum_{j=1}^m \|[A_j(w_2), \partial_x^\alpha] \partial_{x_j} w\|_{L^2} \|\partial_x^\alpha w\|_{L^2} dt$$

$$\leq \int_0^T \sum_{j=1}^m C \|\nabla A_j(w_2)\|_{H^{s-1}} \|\partial_{x_j} w\|_{H^{s-1}} \|\partial_x^\alpha w\|_{L^2} dt$$

$$\leq C'_A N_s(T) \int_0^T \|w_2\|_{H^s} \|\nabla w\|_{H^{s-1}} dt$$

$$\leq \frac{C'_A N_s(T)}{\tau} \int_0^T \|w_2(t)\|_{H^s}^2 dt + \tau C'_A N_s(T) \int_0^T \|\nabla w(t)\|_{H^{s-1}}^2 dt.$$

Thus, summing the relations (95) and (96) for $0 < |\alpha| \leq s$, we get constants $C_2 = C_2(\eta, G, A_1, \dots, A_m) > 0$ and $C''_A = C''_A(A_1, \dots, A_m) > 0$ such that

$$\|w(T)\|_{H^s}^2 + \frac{1}{\tau} \int_0^T \|w_2(t)\|_{H^s}^2 dt \leq C_2 \|w_0\|_{H^s}^2 + \tau C''_A N_s(T)$$

$$\times \int_0^T \|\nabla w(t)\|_{H^{s-1}}^2 dt, \quad (98)$$

when $(C_A + C'_A)N_s(T) \leq 1/2$. The same arguments as in the proof of Theorem 5 lead to

$$\begin{aligned} \frac{N_{*,W_e}}{4\tau^2} \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt &\leq \frac{C_{W_e}}{\tau^2} [\|w(T)\|_{H^s}^2 + \|w_0\|_{H^s}^2] \\ &+ \left(\frac{\|D\|}{\tau} + \frac{2C_{W_e}^2 \|D\|^2}{N_{*,\epsilon} \tau^4} \right) \int_0^T \|w_2\|_{H^s}^2 dt \\ &+ \frac{2C_{W_e}^2 \tilde{C}_{\mathcal{M}}^2}{N_{*,W_e} \tau^2} N_s(T)^2 \int_0^T \|\nabla_x w\|_{H^{s-1}}^2 dt. \end{aligned} \tag{99}$$

Indeed, when one replaces B by B/τ , this leads to replacing N_{*,W_e} by $N_{*,W_e}/\tau^2$, C_{W_e} by C_{W_e}/τ^2 , D by D/τ . Let us introduce

$$\delta = \delta(\tau) := \frac{1}{2} \min \left\{ \frac{\tau^2}{C_{W_e}}, \frac{N_{*,W_e} \tau^3}{\|D\| N_{*,W_e} \tau^3 + 2C_{W_e}^2 \|D\|^2} \right\}. \tag{100}$$

Summing (98) with $\delta \cdot$ (99) we get

$$\begin{aligned} &\|w(T)\|_{H^s} + \frac{1}{\tau} \int_0^T \|w_2(t)\|_{H^s}^2 dt + \frac{\delta N_{*,W_e}}{2\tau^2} \int_0^T \|\nabla w(t)\|_{H^{s-1}}^2 dt \\ &\leq 2(C_2 + 1/2) \|w_0\|_{H^s}^2 + 2 \left(\tau C''_A N_s(T) + \delta \frac{2C_{W_e}^2 \tilde{C}_{\mathcal{M}}^2}{N_{*,W_e} \tau^2} N_s(T)^2 \right) \\ &\quad \times \int_0^T \|\nabla w(t)\|_{H^{s-1}} dt. \end{aligned}$$

So, as long as

$$2\tau C''_A N_s(T) \leq \frac{\delta N_{*,W_e}}{8\tau^2} \quad \text{and} \quad \delta \frac{4C_{W_e}^2 \tilde{C}_{\mathcal{M}}^2}{N_{*,W_e} \tau^2} N_s(T)^2 \leq \frac{\delta N_{*,W_e}}{8\tau^2} \tag{101}$$

we have (94) with $v_2 := 8(C_2 + 1)$. The inequalities (101) are satisfied for every $T > 0$ when

$$\|w_0\| \leq \frac{1}{\sqrt{v_3}} \min \left\{ \frac{\delta N_{*,W_e}}{16\tau^3 C''_A}, \frac{N_{*,W_e}}{32C_{W_e} \tilde{C}_{\mathcal{M}}} \right\}. \tag{102}$$

Thanks to (100), we have

$$\delta(\tau) \sim \frac{N_{*,W_e}}{2C_{W_e}^2 \|D\|^2} \tau^3 \quad \text{when } \tau \rightarrow 0,$$

thus the right-hand side of (102) is bounded from below by a positive constant, for every $\tau \in (0, 1)$. \square

Remark 15. Notice that the dependence of the matrices A_j with respect to w_2 (and not w) is crucial in this proof. With $A_j = A_j(w)$ the same result is an open question.

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