

Null controllability of degenerate parabolic equations of Grushin and Kolmogorov type

K. BEAUCHARD *

Abstract

The goal of this note is to present the results of the references [5] and [4]. We study the null controllability of the parabolic equations associated with the Grushin-type operator $\partial_x^2 + |x|^{2\gamma}\partial_y^2$ ($\gamma > 0$) in the rectangle $(x, y) \in (-1, 1) \times (0, 1)$ or with the Kolmogorov-type operator $v^\gamma\partial_x f + \partial_v^2 f$ ($\gamma \in \{1, 2\}$) in the rectangle $(x, v) \in \mathbb{T} \times (-1, 1)$, under an additive control supported in an open subset ω of the space domain.

We prove that the Grushin-type equation is null controllable in any positive time for $\gamma < 1$ and that there is no time for which it is null controllable for $\gamma > 1$. In the transition regime $\gamma = 1$ and when ω is a strip $\omega = (a, b) \times (0, 1)$, ($0 < a, b \leq 1$), a positive minimal time is required for null controllability.

For the Kolmogorov-type equation with $\gamma = 1$ and periodic-type boundary conditions (in v), we prove that null controllability holds in any positive time, with any control support ω . This improves the previous result [6], in which the control support was a strip $\omega = \mathbb{T} \times (a, b)$.

For the Kolmogorov-type equation with Dirichlet boundary conditions and a strip $\omega = \mathbb{T} \times (a, b)$ ($0 < a < b < 1$) as control support, we prove that null controllability holds in any positive time if $\gamma = 1$, and only in large time if $\gamma = 2$.

Our approach, inspired from [8, 33], is based on 2 key ingredients: the observability of the Fourier components of the solution of the adjoint system (a heat equation with potential), uniformly with respect to the frequency, and the explicit exponential decay rate of these Fourier components.

Key words: null controllability, degenerate parabolic equations, Carleman estimates, hypoelliptic systems.

AMS subject classifications: 35K65, 93B05, 93B07, 34B25

1 Introduction

1.1 Main results

In this article, we consider

*CMLS, Ecole Polytechnique, 91128 Palaiseau Cedex, France, email: Karine.Beauchard@math.polytechnique.fr

- the Grushin-type equations

$$\begin{cases} \partial_t f - \partial_x^2 f - |x|^{2\gamma} \partial_y^2 f = u(t, x, y) 1_\omega(x, y) & (t, x, y) \in (0, \infty) \times \Omega, \\ f(t, x, y) = 0 & (t, x, y) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (1)$$

where $\gamma > 0$, $\Omega := (-1, 1) \times (0, 1)$ and ω is an open subset of Ω ,

- and the Kolmogorov-type equations

$$\partial_t f + v^\gamma \partial_x f - \partial_v^2 f = u(t, x, v) 1_\omega(x, v), \quad (t, x, v) \in (0, +\infty) \times \Omega, \quad (2)$$

where $\gamma \in \{1, 2\}$, $\Omega = \mathbb{T} \times (-1, 1)$ and ω is an open subset of Ω .

For the Kolmogorov-type equations, depending on the value of γ , we use different boundary conditions in variable v : periodic type boundary conditions when $\gamma = 1$

$$\begin{cases} f(t, x - t, -1) = f(t, x + t, +1), & (t, x) \in (0, +\infty) \times \mathbb{T}, \\ \partial_v f(t, x - t, -1) = \partial_v f(t, x + t, +1), & (t, x) \in (0, +\infty) \times \mathbb{T}, \end{cases} \quad (3)$$

or Dirichlet boundary conditions when $\gamma \in \{1, 2\}$

$$f(t, x, -1) = f(t, x, +1) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{T}. \quad (4)$$

We will also use initial data

$$f(0, x, y) = f_0(x, y), \quad (x, y) \in \Omega, \quad (5)$$

for Grushin-type equations and

$$f(0, x, v) = f_0(x, v), \quad (x, v) \in \Omega, \quad (6)$$

for Kolmogorov-type equations.

Let us emphasize that we use the same letter Ω for the spacial domains of both equations, even if they are different.

Both equations are linear control systems in which

- the state is f ,
- the control u is supported in the subset ω ,

and both are degenerate parabolic equations: for the Grushin-type equations, the coefficient of $\partial_y^2 f$ vanishes on the line $\{x = 0\}$, whereas for the Kolmogorov-type equations, the degeneracy happens everywhere (no second derivative with respect to x). We will investigate the null controllability of (1) and (2).

Definition 1 (Null controllability). *Let $T > 0$. System (1) (resp. (2)-(3), resp. (2)-(4)) is null controllable in time T if, for every $f_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \Omega)$ such that the solution of the Cauchy problem (1)-(5) (resp. (2)-(3)-(6), resp. (2)-(4)-(6)) satisfies $f(T, \cdot, \cdot) = 0$.*

System (1) (resp. (2)-(3), resp. (2)-(4)) is null controllable if there exists $T > 0$ such that it is null controllable in time T .

The main results of this paper are stated in Theorems 1 and 2 below. Concerning Grushin-type equations (1), the following null controllability result holds.

Theorem 1. *Let ω be an open subset of $(0, 1) \times (0, 1)$.*

1. *If $\gamma \in (0, 1)$, then system (1)-(5) is null controllable in any time $T > 0$.*
2. *If $\gamma = 1$ and $\omega = (a, b) \times (0, 1)$ where $0 < a < b \leq 1$, then there exists $T^* \geq \frac{a^2}{2}$ such that*
 - *for every $T > T^*$ system (1)-(5) is null controllable in time T ,*
 - *for every $T < T^*$ system (1)-(5) is not null controllable in time T .*
3. *If $\gamma > 1$, then (1)-(5) is not null controllable.*

Concerning Kolmogorov-type equations (2), the following null controllability result holds.

Theorem 2. *1. If $\gamma = 1$ and ω is an open subset of Ω , then the system (2)-(3) is null controllable in any time $T > 0$.*

2. *If $\gamma = 1$ and $\omega = \mathbb{T} \times (a, b)$ with $-1 < a < b < 1$, then the system (2)-(4) is null controllable in any time $T > 0$.*
3. *If $\gamma = 2$ and $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b < 1$ then there exists $T^* \geq a^2/2$ such that*
 - *the system (2)-(4) is null controllable in any time $T > T^*$,*
 - *the system (2)-(4) is not null controllable in time $T < T^*$.*
4. *If $\gamma = 2$ and $\omega = \mathbb{T} \times (a, b)$ with $-1 < a < 0 < b < 1$ then the system (2)-(4) is null controllable in any time $T > 0$.*

Theorem 2 improves the results of reference [6], for system (2)-(3) with $\gamma = 1$ and $\omega = \mathbb{T} \times (a, b)$, $-1 < a < b < 1$. Note that in the third statement, the set $\{v = 0\}$ is not contained in the control location ω contrary to the fourth case.

Theorems 1 and 2 emphasize several behaviors

1. a finite speed of propagation through the set $\{x = 0\}$ for the Grushin-type equations with $\gamma = 1$ and through the set $\{v = 0\}$ for Kolmogorov type equations with $\gamma = 2$ and Dirichlet boundary conditions,
2. a sensitivity to boundary conditions (in v) for the Kolmogorov equation (see the asymptotic behavior of Fourier components in Propositions 10, 13 and 15),

By duality, these null controllability results are equivalent to observability inequalities for the adjoint systems:

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma} \partial_y^2 g = 0 & (t, x, y) \in (0, \infty) \times \Omega, \\ g(t, x, y) = 0 & (t, x, y) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (7)$$

for the Grushin-type equations and

$$\partial_t g - v^\gamma \partial_x g - \partial_v^2 g = 0, \quad (t, x, v) \in (0, +\infty) \times \Omega, \quad (8)$$

for the Kolmogorov-type equations, associated with the following boundary conditions when $\gamma = 1$

$$\begin{cases} g(t, x - T + t, -1) = g(t, x + T - t, 1), & (t, x) \in (0, +\infty) \times \mathbb{T}, \\ \partial_v g(t, x - T + t, -1) = \partial_v g(t, x + T - t, 1), & (t, x) \in (0, +\infty) \times \mathbb{T}, \end{cases} \quad (9)$$

or the following ones for $\gamma \in \{1, 2\}$

$$g(t, x, -1) = g(t, x, 1) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{T}. \quad (10)$$

We will also use initial data

$$g(0, x, y) = g_0(x, y), \quad (x, y) \in \Omega, \quad (11)$$

for (7) or, for (8)

$$g(0, x, v) = g_0(x, v), \quad (x, v) \in \Omega. \quad (12)$$

Definition 2 (Observability). *Let $T > 0$. System (7) (resp. (8)-(9), resp. (8)-(10)) is observable in ω in time T if there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega)$, the solution of the Cauchy-problem (7)-(11) (resp (8)-(9)-(12), resp. (8)-(10)-(12)) satisfies*

$$\int_{\Omega} |g(T, x, y)|^2 dx dy \leq C \int_0^T \int_{\omega} |g(t, x, y)|^2 dx dy dt.$$

System (7) (resp. (8)-(9), resp. (8)-(10)) is observable in ω if there exists $T > 0$ such that it is observable in ω in time T .

For Grushin-type equations (7), the following observability result holds.

Theorem 3. *Let ω be an open subset of $(0, 1) \times (0, 1)$.*

1. *If $\gamma \in (0, 1)$, then system (7) is observable in ω in any time $T > 0$.*
2. *If $\gamma = 1$ and $\omega = (a, b) \times (0, 1)$ where $0 < a < b \leq 1$, then there exists $T^* \geq \frac{a^2}{2}$ such that*
 - *for every $T > T^*$ system (7) is observable in ω in time T ,*
 - *for every $T < T^*$ system (7) is not observable in ω in time T .*
3. *If $\gamma > 1$, then system (7) is not observable in ω .*

Remark 1. When $\gamma = 1$, the geometric restriction on the control domain ω only affects our positive result. Indeed, Theorem 1 trivially implies that (1) fails to be null controllable (if $\gamma = 1$ and T is small) when ω is any connected open set at positive distance from the degeneracy region $\{x = 0\}$. It is also straightforward to observe that, if ω contains a strip containing $\{x = 0\}$, then null controllability holds for any $\gamma > 0$ thanks to standard localization arguments (see [5, Appendix]).

For Kolmogorov-type equations (8), the following observability result holds.

Theorem 4. 1. If $\gamma = 1$ and ω is an open subset of Ω , then the system (8)-(9) is observable in ω in any time $T > 0$.

2. If $\gamma = 1$ and $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b < 1$, then the system (8)-(10) is observable in ω in any time $T > 0$.

3. If $\gamma = 2$ and $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b < 1$, then there exists $T^* \geq a^2/2$ such that

- the system (8)-(10) is observable in ω in any time $T > T^*$,
- the system (8)-(10) is not observable in ω in time $T < T^*$.

4. If $\gamma = 2$ and $\omega = \mathbb{T} \times (a, b)$ with $-1 < a < 0 < b < 1$ then the system (8)-(10) is observable in ω in any time $T > 0$.

1.2 Motivation and bibliographical comments

1.2.1 Null controllability of the heat equation

The null and approximate controllability of the heat equation are essentially well understood subjects for both linear and semilinear equations, for bounded or unbounded domains (see, for instance, [16], [20], [22], [23], [24], [28], [32], [33], [36], [39], [40], [45], [46]) and also with discontinuous (see, e.g. [17], [7], [8], [41]) or singular ([42] and [19]) coefficients.

In particular, the heat equation on a smooth bounded domain Ω of \mathbb{R}^d ($d \in \mathbb{N}^*$), with a source term located on an open subset ω of Ω is null controllable in arbitrarily small time T and with an arbitrarily small control support ω . This result is due, for the case $d = 1$, to H. Fattorini and D. Russell [21, Theorem 3.3], and, for $d \geq 2$, to O. Imanuvilov [30], [31] (see also the book [26] by A. Fursikov and O. Imanuvilov) and G. Lebeau and L. Robbiano [33]. It is then natural to wonder whether the same result holds for degenerate parabolic equations.

1.2.2 Boundary-degenerate parabolic equations

The null controllability of parabolic equations degenerating on the boundary of the domain in one space dimension is well-understood, much less so in higher dimension. Given $0 < a < b < 1$ and $\gamma > 0$, let us consider the 1D equation

$$\partial_t w + \partial_x(x^{2\gamma} \partial_x w) = u(t, x) 1_{(a,b)}(x), \quad (t, x) \in (0, \infty) \times (0, 1),$$

with suitable boundary conditions. Then, null controllability holds if and only if $\gamma \in (0, 1)$ (see [13, 14]), while, for $\gamma \geq 1$, the best result one can show is “regional null controllability” (see [12]), which consists in controlling the solution within the domain of influence of the control. Several extensions of the above results are available in one space dimension, see [1, 37] for equations in divergence form, [11, 10] for nondivergence form operators, and [9, 25] for cascade systems. Fewer results are available for multidimensional problems, mainly in the case of two dimensional parabolic operators which simply degenerate in the normal direction to the boundary of the space domain, see [15].

1.2.3 Parabolic equations degenerating inside the domain

In [38], the authors study linearized Crocco type equations

$$\begin{cases} \partial_t f + \partial_x f - \partial_{vv} f = u(t, x, v) 1_\omega(x, v), & (t, x, v) \in (0, T) \times \mathbb{T} \times (0, 1), \\ f(t, x, 0) = f(t, x, 1) = 0, & (t, x) \in (0, T) \times \mathbb{T}. \end{cases}$$

For a given strict open subset ω of $\mathbb{T} \times (0, 1)$, they prove that null controllability does not hold: the optimal result is regional null controllability. Note that, for Kolmogorov equation (2), the coupling between the diffusion (in v) and the transport (in x at speed v) generates diffusion both in variables x and v (see Propositions 10, 13 and 15).

1.2.4 Hypoellipticity, unique continuation and null controllability

It could be interesting to analyze the connections between null controllability and hypoellipticity.

1.2.5 Hypoellipticity

We recall that a linear differential operator P with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^d$ is called hypoelliptic if, for every distribution u in Ω , u must be a C^∞ function in every open set where so is Pu . The following sufficient condition (which is also essentially necessary) for hypoellipticity is due to Hörmander (see [29]).

Theorem 5. *Let P be a second order differential operator of the form $P = \sum_{j=1}^r X_j^2 + X_0 + c$, where X_0, \dots, X_r denote first order homogeneous differential operators in an open set $\Omega \subset \mathbb{R}^n$ with C^∞ coefficients, and $c \in C^\infty(\Omega)$. Assume that there exists n operators among*

$$X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \dots, [X_{j_1}, [X_{j_2}, [X_{j_3}, [\dots, X_{j_k} \dots]]]],$$

where $j_i \in \{0, 1, \dots, r\}$, which are linearly independent at any given point in Ω . Then, P is hypoelliptic.

Grushin operator $G := \partial_x^2 + |x|^{2\gamma} \partial_y^2$ and Kolmogorov operator $K := v^\gamma \partial_x + \partial_v^2$ are prototypes of hypoelliptic operators of type I ($K = X_0 + X_1^2$) and of type II ($G = X_0^2 + X_1^2$). They are associated to the vector fields

$$X_0(x, v) := \begin{pmatrix} v^\gamma \\ 0 \end{pmatrix}, \quad X_1(x, v) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

that satisfy Hörmander condition for every $\gamma \in \mathbb{N}^*$. Indeed,

$$[X_0, X_1](x, v) = \begin{pmatrix} \gamma v^{\gamma-1} \\ 0 \end{pmatrix}, \quad [X_1, [X_1, X_2]](x, v) = \begin{pmatrix} \gamma(\gamma-1)v^{\gamma-2} \\ 0 \end{pmatrix}.$$

Thus, when $\gamma = 1$, the first iterated Lie bracket is sufficient, whereas when $\gamma = 2$, the second one the required (at $v = 0$), to satisfy Hörmander's condition.

1.2.6 Hypoellipticity and unique continuation

First, let us recall that, in the elliptic case, the unique continuation property is proved by Garofalo in [27] for the Grushin-type operators G and by Alinhac and Zuily in [2] for the Kolmogorov-type operators K . Now, let us focus on the parabolic case.

It is well known that hypoellipticity is not sufficient for unique continuation (see [47]). In particular, Alinhac and Zuily proved in [2] the existence of a C^∞ -zero order perturbation $a(t, x, v)$ such that the operator $\partial_t - v^\gamma \partial_x - \partial_v^2 - a(t, x, v)$ does not satisfy the unique continuation property.

For the Grushin-type equations studied in this article, the unique continuation trivially holds, for every $\gamma > 0$, thanks to the particular geometric configuration (see Proposition 3).

1.2.7 Hypoellipticity and null controllability

Theorems 1 and 2 show that

- for Grushin-type equations, null controllability holds only when the first iterated Lie-bracket is sufficient to satisfy Hörmander's condition ($\gamma \in (0, 1]$),
- whereas for Kolmogorov-type equations, null controllability holds when the two first iterated Lie-brackets are sufficient.

This suggests that a link could relate null controllability of hypoelliptic operators (depending on their type) to the number of iterated Lie brackets that are necessary to satisfy Hörmander's condition. This remains—for the time being—a challenging open problem.

1.3 Structure of the article

In section 2, we prove Theorem 1 for Grushin-type operators. In section 2, we prove Theorem 2 for Kolmogorov-type operators.

2 Proof of Theorem 1 for Grushin-type operators

2.1 Well posedness of the Cauchy-problem, Fourier decomposition and unique continuation

Define the product

$$(f, g) := \int_{\Omega} (f_x g_x + |x|^{2\gamma} f_y g_y) dx dy \quad (13)$$

for every f, g in $C_0^\infty(\Omega)$, and set $V := \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$, where $|f|_V := (f, f)^{1/2}$. Consider the bilinear form a on V defined by

$$a(f, g) = -(f, g) \quad \forall f, g \in V. \quad (14)$$

Moreover, set

$$D(A) := \{f \in V : \exists c > 0 \text{ such that } |a(f, h)| \leq c \|h\|_{L^2(\Omega)} \forall h \in V\}, \quad (15)$$

$$Af := \partial_x^2 f + |x|^{2\gamma} \partial_y^2 f.$$

The following wellposedness result is classical (see, for instance, [35] or [44, Theorem 1.18]).

Proposition 1. *For every $f_0 \in L^2(\Omega)$, $T > 0$ and $u \in L^2(0, T; L^2(\Omega))$, there exists a unique weak solution $f \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; V)$ of the Cauchy problem (1)-(5). This solution satisfies*

$$\|f(t)\|_{L^2(\Omega)} \leq \|f_0\|_{L^2(\Omega)} + \sqrt{T} \|u\|_{L^2(0, T; L^2(\omega))} \quad \forall t \in [0, T]. \quad (16)$$

Moreover, $f(t) \in D(A)$ and $f'(t) \in L^2(\Omega)$ for a.e. $t \in (0, T)$.

Let us consider the weak solution of (7)-(11). Since g belongs to $C([0, T]; L^2(\Omega))$, the function $y \mapsto g(t, x, y)$ belongs to $L^2(0, 1)$ for a.e. $(t, x) \in (0, T) \times (-1, 1)$, thus it can be developed in Fourier series with respect to y as follows

$$g(t, x, y) = \sum_{n \in \mathbb{N}^*} g_n(t, x) \varphi_n(y), \quad (17)$$

where

$$\varphi_n(y) := \sqrt{2} \sin(n\pi y) \quad \forall n \in \mathbb{N}^*$$

and

$$g_n(t, x) := \int_0^1 g(t, x, y) \varphi_n(y) dy \quad \forall n \in \mathbb{N}^*. \quad (18)$$

Proposition 2. *For every $n \geq 1$, g_n is the unique weak solution of*

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2 |x|^{2\gamma} g_n = 0 & (t, x) \in (0, T) \times (-1, 1), \\ g_n(t, \pm 1) = 0 & t \in (0, T), \\ g_n(0, x) = g_{0,n}(x) & x \in (-1, 1), \end{cases} \quad (19)$$

where $g_{0,n} \in L^2(-1, 1)$ is given by $g_{0,n}(x) := \int_0^1 g_0(x, y) \varphi_n(y) dy$.

As a consequence, the parabolic operators of Grushin-type satisfy the unique continuation property

Proposition 3. *Let $T > 0$, $\gamma > 0$, let ω be a bounded open subset of $(0, 1) \times (0, 1)$, and let $g \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; V)$ be a weak solution of (7). If $g \equiv 0$ on $(0, T) \times \omega$, then $g \equiv 0$ on $(0, T) \times \Omega$.*

Proof: Let $\epsilon > 0$ be such that $\omega \subset (\epsilon, 1) \times (0, 1)$. By unique continuation for uniformly parabolic 2D equation, we deduce that $g \equiv 0$ on $(0, T) \times (\epsilon, 1) \times (0, 1)$. Thus, $g_n \equiv 0$ on $(0, T) \times (\epsilon, 1)$ for every $n \in \mathbb{N}^*$. Then, by unique continuation for the uniformly parabolic 1D equation (19), we deduce that $g_n \equiv 0$ on $(0, T) \times (-1, 1)$ for every $n \in \mathbb{N}^*$. \square

2.2 Dissipation speed

Let us introduce, for every $n \in \mathbb{N}^*$, $\gamma > 0$, the operator $A_{n,\gamma}$ defined on $L^2(-1, 1)$ by

$$D(A_{n,\gamma}) := H^2 \cap H_0^1(-1, 1), \quad A_{n,\gamma}\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi. \quad (20)$$

The smallest eigenvalue of $A_{n,\gamma}$ is given by

$$\lambda_{n,\gamma} = \min \left\{ \frac{\int_{-1}^1 [v'(x)^2 + (n\pi)^2|x|^{2\gamma}v(x)^2] dx}{\int_{-1}^1 v(x)^2 dx}; v \in H_0^1(-1, 1), v \neq 0 \right\}. \quad (21)$$

We are interested in the asymptotic behavior (as $n \rightarrow +\infty$) of $\lambda_{n,\gamma}$, which quantifies the dissipation speed of the solution of (19). The following result turns out to be a key point of the proof of Theorem 1; it may be proved with a scaling argument in (21).

Proposition 4. *For every $\gamma > 0$, there are constants $c_* = c_*(\gamma)$, $c^* = c^*(\gamma) > 0$ such that*

$$c_*n^{\frac{2}{1+\gamma}} \leq \lambda_{n,\gamma} \leq c^*n^{\frac{2}{1+\gamma}} \quad \forall n \in \mathbb{N}^*.$$

2.3 Proof of the negative statements of Theorem 3

The goal of this section is the proof of the following results.

- if $\gamma = 1$, $\omega \subset (a, 1) \times (0, 1)$ for some $a > 0$ and $T < \frac{a^2}{2}$, then system (8) is not observable in ω in time T ,
- if $\gamma > 1$ and $T > 0$, then system (8) is not observable in ω in time T .

Without loss of generality, one may assume that $\omega = (a, b) \times (0, 1)$ with $0 < a < b < 1$.

2.3.1 Strategy for the proof

Let g be the solution of (7)-(11). Then, g can be represented as in (17), and we emphasize that, for a.e. $t \in (0, T)$ and every $-1 \leq a_1 < b_1 \leq 1$,

$$\int_{(a_1, b_1) \times (0, 1)} |g(t, x, y)|^2 dx dy = \sum_{n=1}^{\infty} \int_{a_1}^{b_1} |g_n(t, x)|^2 dx$$

(Bessel-Parseval equality). Thus, in order to prove Theorem 3, it is sufficient to study the observability of system (19) uniformly with respect to $n \in \mathbb{N}^*$.

Definition 3 (Uniform observability). *Let $0 < a < b \leq 1$ and $T > 0$. System (19) is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$ if there exists $C > 0$ such that, for every $n \in \mathbb{N}^*$, $g_{0,n} \in L^2(-1, 1)$, the solution of (19) satisfies*

$$\int_{-1}^1 |g_n(T, x)|^2 dx \leq C \int_0^T \int_a^b |g_n(t, x)|^2 dx.$$

System (19) is observable in (a, b) uniformly with respect to $n \in \mathbb{N}^$ if there exists $T > 0$ such that it is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*

The negative parts of the conclusion of Theorem 3 follow from the result below.

Theorem 6. *Let $0 < a < b \leq 1$.*

1. *If $\gamma = 1$ and $T < \frac{a^2}{2}$, then system (19) is not observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*
2. *If $\gamma > 1$, then system (19) is not observable in (a, b) uniformly with respect to $n \in \mathbb{N}^*$.*

The proof of Theorem 6 relies on the use of appropriate test functions that falsify uniform observability. This is proved thanks to a well adapted maximum principle (see Lemma 1) and explicit supersolutions (see (25)) for $\gamma > 1$, and thanks to direct computations for $\gamma = 1$.

2.3.2 Proof of Theorem 6 for $\gamma > 1$

Let $\gamma \in [1, +\infty)$ be fixed and $T > 0$. For every $n \in \mathbb{N}^*$, we denote by λ_n (instead of $\lambda_{n,\gamma}$) the first eigenvalue of the operator $A_{n,\gamma}$ defined in Section 2.2, and by v_n the associated positive eigenvector of norm one, which satisfies

$$\begin{cases} -v_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n]v_n(x) = 0, & x \in (-1, 1), \quad n \in \mathbb{N}^*, \\ v_n(\pm 1) = 0, \quad v_n \geq 0, \\ \|v_n\|_{L^2(-1,1)} = 1. \end{cases}$$

Then, for every $n \geq 1$, the function

$$g_n(t, x) := v_n(x)e^{-\lambda_n t} \quad \forall (t, x) \in \mathbb{R} \times (-1, 1),$$

solves the adjoint system (19). Let us note that

$$\begin{aligned} \int_{-1}^1 g_n(T, x)^2 dx &= e^{-2\lambda_n T}, \\ \int_0^T \int_a^b g_n(t, x)^2 dx dt &= \frac{1 - e^{-2\lambda_n T}}{2\lambda_n} \int_a^b v_n(x)^2 dx. \end{aligned}$$

So, in order to prove that uniform observability fails, it suffices to show that

$$\frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx \rightarrow 0 \text{ when } n \rightarrow +\infty. \quad (22)$$

The above convergence will be obtained comparing v_n with an explicit supersolution of the problem on a suitable subinterval of $[-1, 1]$, thanks to the following maximum principle

Lemma 1. *Let $0 < a < b < 1$. For every $n \in \mathbb{N}^*$, set*

$$x_n := \left(\frac{\lambda_n}{(n\pi)^2} \right)^{\frac{1}{2\gamma}} \quad (23)$$

and let $W_n \in C^2([x_n, 1], \mathbb{R})$ be a solution of

$$\begin{cases} -W_n''(x) + [(n\pi)^2 x^{2\gamma} - \lambda_n]W_n(x) \geq 0, & x \in (x_n, 1), \\ W_n(1) \geq 0, \\ W_n'(x_n) < -\sqrt{x_n} \lambda_n. \end{cases} \quad (24)$$

Then there exists $n_* \in \mathbb{N}^*$ such that, for every $n \geq n_*$,

$$\int_a^b v_n(x)^2 dx \leq \int_a^b W_n(x)^2 dx.$$

In order to apply Lemma 1, we need an explicit supersolution W_n of (24) of the form

$$W_n(x) = C_n e^{-\mu_n x^{\gamma+1}}, \quad (25)$$

where $C_n, \mu_n > 0$. Notice that, in particular, $W_n(1) \geq 0$.

First step: let us prove that, for an appropriate choice of μ_n , the first inequality of (24) holds. Since

$$W_n'(x) = -\mu_n(\gamma+1)x^\gamma W_n(x),$$

$$W_n''(x) = [-\mu_n\gamma(\gamma+1)x^{\gamma-1} + \mu_n^2(\gamma+1)^2x^{2\gamma}]W_n(x),$$

the first inequality of (24) holds if and only if, for every $x \in (x_n, 1)$,

$$[(n\pi)^2 - \mu_n^2(\gamma+1)^2]x^{2\gamma} + \mu_n\gamma(\gamma+1)x^{\gamma-1} \geq \lambda_n. \quad (26)$$

In particular, it holds when

$$\mu_n \leq \frac{n\pi}{\gamma+1} \quad (27)$$

and

$$[(n\pi)^2 - \mu_n^2(\gamma+1)^2]x_n^{2\gamma} + \mu_n\gamma(\gamma+1)x_n^{\gamma-1} \geq \lambda_n. \quad (28)$$

Indeed, in this case, the left hand side of (26) is an increasing function of x . In view of (23), and after several simplifications, inequality (28) can be recast as

$$\mu_n \leq \frac{\gamma}{\gamma+1} \left(\frac{(n\pi)^2}{\lambda_n} \right)^{\frac{1}{2} + \frac{1}{2\gamma}}.$$

So, recalling (27), in order to satisfy the first inequality of (24) we can take

$$\mu_n := \min \left\{ \frac{n\pi}{\gamma+1}; \frac{\gamma}{\gamma+1} \left(\frac{(n\pi)^2}{\lambda_n} \right)^{\frac{1}{2} + \frac{1}{2\gamma}} \right\}. \quad (29)$$

For the following computations, it is important to notice that, thanks to (29) and Proposition 4, for n large enough μ_n is of the form

$$\mu_n = C_1(\gamma)n. \quad (30)$$

Second step: let us prove that, for an appropriate choice of C_n , the third inequality of (24) holds. Since

$$W_n'(x_n) = -C_n\mu_n(\gamma+1)x_n^\gamma e^{-\mu_n x_n^{\gamma+1}},$$

the third inequality of (24) is equivalent to

$$C_n > \frac{\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-\frac{1}{2}}}.$$

Therefore, it is sufficient to choose

$$C_n := \frac{2\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-\frac{1}{2}}}. \quad (31)$$

Third step: let us prove condition (22). Thanks to Lemma 1, (25), (30) and (31), for every $n \geq n_*$,

$$\begin{aligned} \frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx &\leq \frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b W_n(x)^2 dx \leq \frac{e^{2\lambda_n T}}{\lambda_n} W_n(a)^2 \\ &\leq \frac{e^{2\lambda_n T}}{\lambda_n} C_n^2 e^{-2\mu_n a^{1+\gamma}} \leq \frac{e^{2\lambda_n T}}{\lambda_n} \frac{4\lambda_n^2 e^{2\mu_n x_n^{\gamma+1}}}{(\gamma+1)^2 \mu_n^2 x_n^{2\gamma-1}} e^{-2\mu_n a^{1+\gamma}}. \end{aligned}$$

By identities (23), (30) and Proposition 4, we have

$$\mu_n x_n^{\gamma+1} \leq C_2(\gamma) \quad \forall n \in \mathbb{N}^*,$$

thus

$$\frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx \leq e^{2n(\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma})} \frac{4\lambda_n e^{2C_2(\gamma)}}{(\gamma+1)^2 \mu_n^2 x_n^{2\gamma-1}}. \quad (32)$$

Since $\gamma > 1$, we deduce from Proposition 4 that

$$\frac{\lambda_n}{n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

So, for every $T > 0$, there exists $n_\# \geq n_*$ such that, for every $n \geq n_\#$,

$$\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma} < -\frac{1}{2}C_1(\gamma)a^{1+\gamma}. \quad (33)$$

Then, inequality (32) yields condition (22) (since the term that multiplies the exponential behaves like a rational fraction of n).

2.3.3 Proof of Theorem 6 for $\gamma = 1$

In this section, we take $\gamma = 1$ and keep the abbreviated forms λ_n , v_n for $\lambda_{n,\gamma}$, $v_{n,\gamma}$ introduced in Section 2.2. When $T < \frac{a^2}{2}$, we can easily deduce from the following lemma that (22) holds; thus, system (19) is not observable in (a, b) uniformly with respect to $n \in \mathbb{N}^*$.

Lemma 2. *Let a and b be real numbers such that $0 < a < b \leq 1$. Then*

$$\lambda_n \sim n\pi \quad (34)$$

and

$$\int_a^b v_n(x)^2 dx \sim \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}}. \quad (35)$$

as $n \rightarrow +\infty$.

This Lemma is proved by approximating $v_n(x)$ thanks to the Gaussian function $G(x) := e^{-\frac{x^2}{2}}$, which is the first eigenvector of $-\partial_x^2 + x^2$ on the real line.

2.4 Proof of the positive statements of Theorem 1

The goal of this section is the proof of the following results:

- if $\gamma \in (0, 1)$, then system (1) is null controllable in any time $T > 0$,
- if $\gamma = 1$ and $\omega = (a, b) \times (0, 1)$, with $0 < a < b \leq 1$, then there exists $T_1 > 0$ such that system (1) is null controllable in any time $T > T_1$ or, equivalently, system (7) is observable in ω in any time $T > T_1$.

The proof of these results relies on a new global Carleman estimate for solutions of (19), stated in the next section.

2.4.1 A global Carleman estimate

For $n \in \mathbb{N}^*$, we introduce the operator

$$\mathcal{P}_n g := \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2 |x|^{2\gamma} g.$$

Proposition 5. *Let $\gamma \in (0, 1]$ and let $a, b \in \mathbb{R}$ be such that $0 < a < b \leq 1$. Then there exist a weight function $\beta \in C^1([-1, 1]; \mathbb{R}_+^*)$ and positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that for every $n \in \mathbb{N}^*$, $T > 0$, and $g \in C^0([0, T]; L^2(-1, 1)) \cap L^2(0, T; H_0^1(-1, 1))$ the following inequality holds*

$$\begin{aligned} & \mathcal{C}_1 \int_0^T \int_{-1}^1 \left(\frac{M}{t(T-t)} \left| \frac{\partial g}{\partial x}(t, x) \right|^2 + \frac{M^3}{(t(T-t))^3} |g(t, x)|^2 \right) e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \\ & \leq \int_0^T \int_{-1}^1 |\mathcal{P}_n g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt + \int_0^T \int_a^b \frac{M^3}{(t(T-t))^3} |g(t, x)|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dx dt \end{aligned} \quad (36)$$

where $M := \mathcal{C}_2 \max\{T + T^2; nT^2\}$.

Remark 2. In the case of $\gamma \in [1/2, 1]$, our weight β will be the classical one. On the other hand, for $\gamma \in (0, 1/2)$ we follow the strategy of [1, 11, 37], adapting the weight β to the nonsmooth coefficient $|x|^{2\gamma}$.

2.4.2 Uniform observability

The Carleman estimate of Proposition 5 allows to prove the following uniform observability result.

Proposition 6. *Let $\gamma \in (0, 1)$ and let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$. Then there exists $C > 0$ such that for every $T > 0$, $n \in \mathbb{N}^*$, and $g_{0,n} \in L^2(-1, 1)$ the solution of (19) satisfies*

$$\int_{-1}^1 g_n(T, x)^2 dx \leq T^2 e^{C \left(1 + T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_a^b g_n(t, x)^2 dx dt.$$

Let us recall that explicit bounds on the observability constant of the heat equation with a potential are already known, but not sufficient in our situation (see, for instance [23, Theorem 1.3], [16, Theorem 2.3], [18]).

Proof of Proposition 6: We derive an explicit observability constant from the Carleman estimate of Proposition 5. For $t \in (T/3, 2T/3)$, we have

$$\frac{4}{T^2} \leq \frac{1}{t(T-t)} \leq \frac{9}{2T^2}$$

and

$$\int_{-1}^1 g(T, x)^2 dx \leq \int_{-1}^1 g(t, x)^2 dx e^{-\lambda_n \frac{T}{3}}.$$

Thus,

$$\mathcal{C}_1 \frac{64M^3}{T^6} e^{-\frac{9M\beta^*}{2T^2} \frac{T}{3}} e^{\lambda_n \frac{T}{3}} \int_{-1}^1 g(T, x)^2 dx \leq \mathcal{C}_3 \int_0^T \int_a^b g(t, x)^2 dx dt$$

where $\beta^* := \max\{\beta(x) : x \in [-1, 1]\}$, $\beta_* := \min\{\beta(x) : x \in [-1, 1]\}$ and $\mathcal{C}_3 := \max\{x^3 e^{-\beta_* x}\}$. Using the inequality $M \geq \mathcal{C}_2[T + T^2]$ and Proposition 4, we get

$$\int_{-1}^1 g(T, x)^2 dx \leq \mathcal{C}_4 T^2 e^{c_1 \frac{M}{T^2} - c_2 n^{\frac{2}{1+\gamma}} T} \int_0^T \int_a^b g(t, x)^2 dx dt \quad (37)$$

for some constants $c_1, c_2, \mathcal{C}_4 > 0$ (independent of n, T and g).

First case: $n < 1 + \frac{1}{T}$. Then, $M = \mathcal{C}_2(T + T^2)$ thus

$$\int_{-1}^1 g(T, x)^2 dx \leq \mathcal{C}_4 T^2 e^{c_1 \mathcal{C}_2 (1 + \frac{1}{T})} \int_0^T \int_a^b g(t, x)^2 dx dt.$$

Second case: $n \geq 1 + \frac{1}{T}$. Then, $M = \mathcal{C}_2 n T^2$. The maximum value of the function $x \mapsto c_1 \mathcal{C}_2 x - c_2 x^{\frac{2}{1+\gamma}} T$ on $(0, +\infty)$ is of the form $c_3 T^{-\frac{1+\gamma}{1-\gamma}}$ for some constant $c_3 > 0$ (independent of T). Thus,

$$\int_{-1}^1 g(T, x)^2 dx \leq \mathcal{C}_4 T^2 e^{c_3 T^{-\frac{1+\gamma}{1-\gamma}}} \int_0^T \int_a^b g(t, x)^2 dx dt.$$

This gives the conclusion. \square

In the case of $\gamma = 1$, we also have the following result.

Proposition 7. *Assume $\gamma = 1$. Let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$. Then there exists $T_1 > 0$ such that, for every $T > T_1$, system (19) is observable in (a, b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*

Proof of Proposition 7: One can follow the lines of the previous proof until (37). Then, for $n \geq 1 + \frac{1}{T}$, we have $M = \mathcal{C}_2 n T^2$. Thus,

$$\int_{-1}^1 g(T, x)^2 dx \leq \mathcal{C}_4 T^2 e^{[c_1 \mathcal{C}_2 - c_2 T] n} \int_0^T \int_a^b g(t, x)^2 dx dt.$$

This proves Proposition 7 with $T_1 := c_1 \mathcal{C}_2 / c_2$. \square

2.4.3 Construction of the control function for $\gamma \in (0, 1)$

The goal of this section is the proof of null controllability in any time $T > 0$ for $\gamma \in (0, 1)$. Our construction of the control steering the initial state to zero is the one of [8], which is in turn inspired by [33] (see also [34]).

For $n \in \mathbb{N}^*$, we define $\varphi_n(y) := \sqrt{2} \sin(n\pi y)$ and $H_n := L^2(-1, 1) \otimes \varphi_n$, which is a closed subspace of $L^2(\Omega)$. For $j \in \mathbb{N}$, we define $E_j := \bigoplus_{n \leq 2^j} H_n$ and denote by Π_{E_j} the orthogonal projection onto E_j .

Proposition 8. Let $\gamma \in (0, 1)$, and let $a, b, c, d \in \mathbb{R}$ be such that $0 < a < b < 1$ and $0 < c < d < 1$. Then there exists a constant $C > 0$ such that for every $T > 0$, every $j \in \mathbb{N}^*$, and every $g_0 \in E_j$ the solution of (8) satisfies

$$\int_{\Omega} g(T, x, y)^2 dx dy \leq T^2 e^{C \left(2^j + T^{-\frac{1+\gamma}{1-\gamma}} \right)} \int_0^T \int_{\omega} g(t, x, y)^2 dx dy dt$$

where $\omega := (a, b) \times (c, d)$.

For the proof of Proposition 8 we shall need the following inequality obtained in [33] (see also [34]).

Proposition 9. Let $c, d \in \mathbb{R}$ be such that $c < d$. There exists $C > 0$ such that, for every $L \in \mathbb{N}^*$ and $(b_k)_{1 \leq k \leq L} \in \mathbb{R}^L$,

$$\sum_{k=1}^L |b_k|^2 \leq e^{CL} \int_c^d \left| \sum_{k=1}^L b_k \varphi_k(y) \right|^2 dy.$$

Proof of Proposition 8: Let $(g_{0,n})_{1 \leq n \leq 2^j} \in L^2(-1, 1)^{2^j}$ be such that

$$g_0(x, y) = \sum_{n=1}^{2^j} g_{0,n}(x) \varphi_n(y).$$

Then the solution of (8) is given by

$$g(t, x, y) = \sum_{n=1}^{2^j} g_n(t, x) \varphi_n(y)$$

where, for every $n \in \mathbb{N}^*$, g_n is the solution of (19). Applying Propositions 6 and 9, and recalling that $(\varphi_n)_{n \in \mathbb{N}^*}$ is an orthonormal sequence of $L^2(0, 1)$, we deduce

$$\begin{aligned} \int_{\Omega} g(T, x, y)^2 dx dy &= \sum_{n=1}^{2^j} \int_{-1}^1 g_n(T, x)^2 dx \\ &\leq T^2 e^{C \left(1 + T^{-\frac{1+\gamma}{1-\gamma}} \right)} \sum_{n=1}^{2^j} \int_0^T \int_a^b g_n(t, x)^2 dx dt \\ &\leq T^2 e^{C \left(2^j + T^{-\frac{1+\gamma}{1-\gamma}} \right)} \int_0^T \int_a^b \int_c^d \left| \sum_{n=1}^{2^j} g_n(t, x) \varphi_n(y) \right|^2 dy dx dt \\ &= T^2 e^{C \left(2^j + T^{-\frac{1+\gamma}{1-\gamma}} \right)} \int_0^T \int_{\omega} g(t, x, y)^2 dx dy dt, \end{aligned}$$

where the constant C may change from line to line. \square

Let $T > 0$ and $f_0 \in L^2(\Omega)$. We now proceed to construct a control $u \in L^2(0, T; L^2(\Omega))$ such that the solution of (1)-(5) satisfies $f(T, \cdot) \equiv 0$. Fix $\rho \in \mathbb{R}$ with

$$0 < \rho < \frac{1-\gamma}{1+\gamma} \tag{38}$$

and let $K = K(\rho) > 0$ be such that $K \sum_{j=1}^{\infty} 2^{-j\rho} = T$. Let $(a_j)_{j \in \mathbb{N}}$ be defined by

$$\begin{cases} a_0 = 0 \\ a_{j+1} = a_j + 2T_j, \quad j \geq 0, \end{cases}$$

where $T_j := K2^{-j\rho}$ for every $j \in \mathbb{N}$. We now define the control u in the following way. On $[a_j, a_j + T_j]$, we apply a control u such that $\Pi_{E_j} f(a_j + T_j, \cdot) = 0$ and

$$\|u\|_{L^2(a_j, a_j + T_j; L^2(\Omega))} \leq \mathcal{C}_j \|f(a_j, \cdot)\|_{L^2(\Omega)}$$

where, in view of Proposition 8,

$$\mathcal{C}_j := e^{C(2^j + T_j^{-\frac{1+\gamma}{1-\gamma}})}.$$

Observe that, in light of (16),

$$\|f(a_j + T_j, \cdot)\|_{L^2(\Omega)} \leq (1 + \sqrt{T_j} \mathcal{C}_j) \|f(a_j, \cdot)\|_{L^2(\Omega)}.$$

Then, on the interval $[a_j + T_j, a_{j+1}]$ we apply no control in order to take advantage of the natural exponential decay of the solution, thus obtaining

$$\|f(a_{j+1}, \cdot)\|_{L^2(\Omega)} \leq e^{-\lambda_{2^j} T_j} \|f(a_j + T_j, \cdot)\|_{L^2(\Omega)},$$

where λ_n is defined in (21). Combining the above inequalities, we conclude that

$$\|f(a_{j+1}, \cdot)\|_{L^2(\Omega)} \leq \exp\left(\sum_{k=1}^{2^j} [\ln(1 + \sqrt{T_k} \mathcal{C}_k) - C(2^k)^{\frac{2}{1+\gamma}} T_k]\right) \|f_0\|_{L^2(\Omega)}.$$

The choice of ρ ensures that the sum in the exponential diverges to $-\infty$ as $j \rightarrow +\infty$, forcing $f(T, \cdot) \equiv 0$. The fact that $u \in L^2(0, T; L^2(\Omega))$ can be checked by similar arguments. \square

2.4.4 End of the proof of Theorems 1 and 3

Let ω be an open subset of $(0, 1) \times (0, 1)$. There exists $a, b, c, d \in \mathbb{R}$ with $0 < a < b < 1$, $0 < c < d < 1$ such that $(a, b) \times (c, d) \subset \omega$.

The first (resp. third) statement of Theorem 3 has been proved in Section 2.4.3 (resp. Section 2.3); let us prove the second one.

Let us consider $\gamma = 1$ and $\omega = (a, b) \times (0, 1)$. From Proposition 7, we deduce that (7) is observable in ω in any time $T > T_1$. From Theorem 6, we deduce that for any time $T < \frac{a^2}{2}$, (7) is not observable in ω in time T . Thus, the quantity

$$T^* := \inf\{T > 0; \text{system (7) is observable in } \omega \text{ in time } T\}$$

is well defined and belongs to $[\frac{a^2}{2}, +\infty)$. Clearly, observability in some time T_{\sharp} implies observability in any time $T > T_{\sharp}$, so

- for every $T > T^*$, (8) is observable in ω in time T ,
- for every $T < T^*$, (8) is not observable in ω in time T .

3 Proof of Theorem 2 for Kolmogorov-type operators

3.1 With $\gamma = 1$ and periodic-type boundary conditions

First, let us recall the following well posedness result, for the Cauchy-problem (2)-(3).

Proposition 10. *Let $T > 0$, $f_0 \in L^2(\Omega)$ and $u \in L^2((0, T) \times \Omega)$. There exists a unique solution $f \in C^0([0, T], L^2(\Omega))$ of the Cauchy problem (2)-(3)-(6). Moreover, if $u \equiv 0$, the Fourier components*

$$f_n(t, v) := \int_{\mathbb{T}} f(t, x, v) e^{-inx} dx, \quad t \in (0, +\infty), v \in (-1, 1), n \in \mathbb{Z}$$

satisfy

$$\|f_n(t, \cdot)\|_{L^2(-1, 1)} \leq \|f_n(0, \cdot)\|_{L^2(-1, 1)} e^{-\frac{n^2 t^3}{12}}, \quad \forall t > 0, n \in \mathbb{Z}.$$

The proof of Theorem 2 for Kolmogorov equation with $\gamma = 1$ and periodic-type boundary conditions is the same as the proof of Theorem 1 for Grushin equation with $\gamma \in (0, 1)$. There are 2 key points. The first one is the explicit exponential rate of the Fourier components emphasized in the previous lemma. The second one is the following global Carleman estimate for the operator

$$\mathcal{P}_{n, \gamma} g := \partial_t g + inv^\gamma g - \partial_v^2 g, \quad n \in \mathbb{Z}, \quad \gamma \in \mathbb{N}^*.$$

Proposition 11. *We assume $\gamma \in \mathbb{N}^*$ (resp. $\gamma = 1$). Let a, b be such that $-1 < a < b < 1$. There exist a weight function $\beta \in C^1([-1, 1], \mathbb{R}_+^*)$, positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that, for every $n \in \mathbb{Z}$, $\gamma \in \{1, 2\}$, $T > 0$ and $g \in C^0([0, T], L^2(-1, 1)) \cap L^2(0, T; H_0^1(-1, 1))$ (resp. $g \in C^0([0, T], L^2(-1, 1)) \cap L^2(0, T; H^1(-1, 1))$) such that $g(t, -1) = g(t, 1)e^{i2n(T-t)}$ and $\partial_v g(t, -1) = \partial_v g(t, 1)e^{i2n(T-t)}$ the following inequality holds*

$$\begin{aligned} & \mathcal{C}_1 \int_0^T \int_{-1}^1 \left(\frac{M}{t(T-t)} \left| \frac{\partial g}{\partial v}(t, v) \right|^2 + \frac{M^3}{(t(T-t))^3} |g(t, v)|^2 \right) e^{-\frac{M\beta(v)}{t(T-t)}} dv dt \\ & \leq \int_0^T \int_{-1}^1 |\mathcal{P}_{n, \gamma} g|^2 e^{-\frac{M\beta(v)}{t(T-t)}} dv dt + \int_0^T \int_a^b \frac{M^3}{(t(T-t))^3} |g(t, v)|^2 e^{-\frac{M\beta(v)}{t(T-t)}} dv dt \end{aligned} \quad (39)$$

where $M := \mathcal{C}_2 \max\{T + T^2; \sqrt{|n|}T^2\}$.

The proof of this estimate is classical (see [26]): our weight β is the usual one. We only track carefully the behavior with respect to n of the different constants.

Then, for the construction of the control function, one may conclude with a parameter ρ such that

$$0 < \rho < \frac{1}{3}.$$

This approach works because the dissipation speed (n^2) in Proposition 10 is stronger than the cost ($\sqrt{|n|}$) provided by Proposition 5, and also stronger than the constant (n) in the Lebeau-Robbiano Lemma (see Proposition 9).

3.2 With $\gamma = 2$ and Dirichlet boundary conditions

In this paragraph, $\gamma \in \{1, 2\}$. Let

$$\mathcal{V} := \{f \in C^\infty(\mathbb{T} \times (-1, 1)); \exists K \subset (-1, 1) \text{ compact s.t. } \text{Supp}(f) \subset \mathbb{T} \times K\}.$$

For $f \in \mathcal{V}$, we define

$$|f|_V := \left(\int_{\Omega} |\partial_v f(x, v)|^2 dx dv \right)^{1/2}$$

and $V := \text{Adh}_{|\cdot|_V}(\mathcal{V})$. We define the operator A_γ by

$$\begin{aligned} D(A_\gamma) &:= \{f \in V; -\partial_v^2 f + v^\gamma \partial_x f \in L^2(\Omega)\}, \\ A_\gamma f &:= -\partial_v^2 f + v^\gamma \partial_x f. \end{aligned}$$

First, let us recall the following well posedness result.

Proposition 12. *Let $\gamma \in \{1, 2\}$. For every $T > 0$, $u \in L^2((0, T) \times \Omega)$, $f_0 \in L^2(\Omega)$ there exists a unique weak solution $f \in C^0([0, T], L^2(\Omega)) \cap L^2((0, T), V)$ of (2)-(4)-(6). Moreover, $f(t) \in D(A_\gamma)$ and $\partial_t f(t) \in L^2(\Omega)$ for a.e. $t \in (0, T)$.*

The proof of Theorem 2 for the Kolmogorov equation with $\gamma = 2$, Dirichlet boundary conditions

- and $a < 0 < b$ may be proved with a classical cut-off argument (see [6] for more details),
- and $0 < a < b < 1$ is the same as the one of Theorem 1 for Grushin-type equations with $\gamma = 1$.

Here, we only state the key point in the proof of the positive result when $0 < a < b < 1$.

Proposition 13. *We assume $\gamma = 2$. There exists $K, \delta > 0$ such that, for every $n \in \mathbb{Z} - \{0\}$ and $g_{0,n} \in H^1(-1, 1)$, the solution of*

$$\begin{cases} \partial_t g_n - inv^\gamma g_n - \partial_v^2 g_n = 0, & (t, v) \in (0, +\infty) \times (-1, 1), \\ g_n(t, \pm 1) = 0, & t \in (0, +\infty), \\ g_n(0, v) = g_{0,n}(v), & v \in (-1, 1), \end{cases} \quad (40)$$

satisfies

$$\int_{-1}^1 |g_n(t, v)|^2 dv \leq K e^{-\delta \sqrt{|n|}t} \int_{-1}^1 \left(\frac{1}{\sqrt{|n|}} |\partial_v g_{0,n}(v)|^2 + \sqrt{|n|} |v g_{0,n}(v)|^2 \right) dv, \forall t > 0.$$

This Proposition is proved with strict Lyapunov functions inspired from [43]. Note that this statement allows to prove Theorem 4 with $\gamma = 2$ and $\omega = \mathbb{T} \times (a, b)$, $0 < a < b < 1$ because the dissipation $\sqrt{|n|}T$ (in Proposition 13) is stronger than the cost $\sqrt{|n|}$ (in Proposition 11) in time T large enough. However, it is not stronger than the constant $(|n|)$ of Lebeau-Robbiano's Lemma, thus we cannot conclude with an arbitrary control location ω .

The proof of Proposition 13 relies on the following result.

Proposition 14. *There exists $A, B, C, \delta > 0$ with $B^2 < AC$ such that, for every $L > 0$ and $h_0 \in H^1(-L, L)$, the solution of*

$$\begin{cases} \partial_\tau h = \partial_y^2 h + iy^2 h, & (\tau, y) \in (0, +\infty) \times (-L, L), \\ h(\tau, \pm L) = 0, & \tau \in (0, +\infty), \\ h(0, y) = h_0(y), & y \in (-L, L), \end{cases} \quad (41)$$

satisfies

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\delta t}, \forall t > 0, \quad (42)$$

where

$$\mathcal{L}(\tau) = \int_{-L}^L \left(|h(\tau, y)|^2 + A|\partial_y h(\tau, y)|^2 - 2B\Im[\overline{y h(\tau, y)} \partial_y h(\tau, y)] + C|y h(\tau, y)|^2 \right) dy.$$

Proof of Proposition 14: This proof is inspired from [43]. Let $A, B, C > 0$ be such that

$$B^2 < AC \quad \text{and} \quad A^2 + C^2 < \frac{B}{2} \quad (43)$$

(for instance $A = \epsilon \tilde{A}$, $B = \epsilon \tilde{B}$, $C = \epsilon \tilde{C}$ for any $\tilde{A}, \tilde{B}, \tilde{C}, \epsilon > 0$ such that $\tilde{B}^2 < \tilde{A} \tilde{C}$ and $\epsilon(\tilde{A}^2 + \tilde{C}^2) < \tilde{B}/2$). Easy computations give

$$\begin{aligned} \frac{1}{2} \frac{d\mathcal{L}}{d\tau} = & -3B\|yh\|^2 - \|\partial_y h\|^2 - C\|y\partial_y h\|^2 - A\|\partial_y^2 h\|^2 \\ & + C\|h\|^2 - 2A\Im \left[\int_{-L}^L y \partial_y \bar{h} h \right] - 2B\Im \left[\int_{-L}^L y \partial_y^2 \bar{h} \partial_y h \right]. \end{aligned}$$

Thanks to the following inequalities

$$\begin{aligned} C\|h\|^2 & \leq 2C\|yh\| \|\partial_y h\| \leq \frac{B}{2}\|yh\|^2 + \frac{2C^2}{B}\|\partial_y h\|^2, \\ -2A\Im \left[\int_0^L y \partial_y \bar{h} h \right] & \leq \frac{B}{2}\|yh\|^2 + \frac{2A^2}{B}\|\partial_y h\|^2, \\ -2B\Im \left[\int_0^L y \partial_y^2 \bar{h} \partial_y h \right] & \leq A\|\partial_y^2 h\|^2 + \frac{B^2}{A}\|y\partial_y h\|^2, \end{aligned}$$

we get

$$\frac{1}{2} \frac{d\mathcal{L}}{d\tau} \leq -2B\|yh\|^2 - \left(1 - \frac{2(A^2 + C^2)}{B} \right) \|\partial_y h\|^2.$$

Thanks to (43), there exists $\delta > 0$ (independent of L) such that $\frac{d\mathcal{L}}{d\tau} \leq -\delta\mathcal{L}$, which gives the conclusion. \square

Proof of Proposition 13: One may assume that $n > 0$, otherwise, consider \bar{g}_n . In order to simplify the notations, we write g , instead of g_n . The function $h(\tau, y)$ defined by

$$g(t, v) = h(\sqrt{n}t, \sqrt[4]{n}v)$$

satisfies (41) with $L = \sqrt[4]{n}$ and $h_0(y) := g_{0,n}(y/\sqrt[4]{n})$. From the previous proposition, we know that

$$\tilde{\mathcal{L}}(t) = \int_{-1}^1 \left(|g(t, v)|^2 + \frac{A}{\sqrt{n}} |\partial_v g(t, v)|^2 - 2B\Im[\overline{vg(t, v)} \partial_v g(t, v)] + C\sqrt{n}|vg(t, v)|^2 \right) dv$$

satisfies $\tilde{\mathcal{L}}(t) \leq \tilde{\mathcal{L}}(0)e^{-\delta\sqrt{n}t}$. Moreover, using (43) and

$$\|g\|^2 \leq 2\|vg\|\|\partial_v g\| \leq \sqrt{n}\|vg\|^2 + \frac{1}{\sqrt{n}}\|\partial_v g\|^2$$

we get

$$\tilde{\mathcal{L}}(0) \leq \int_{-1}^1 \left(\frac{2A+1}{\sqrt{n}} |\partial_v g_0(v)|^2 + (2C+1)\sqrt{n}|vg_0(v)|^2 \right) dv.$$

Thus

$$\begin{aligned} \int_{-1}^1 |g_n(t, v)|^2 dv &\leq \tilde{\mathcal{L}}(t) \\ &\leq \tilde{\mathcal{L}}(0)e^{-\delta\sqrt{n}t} \\ &\leq K \int_{-1}^1 \left(\frac{1}{\sqrt{n}} |\partial_v g_0(v)|^2 + \sqrt{n}|vg_0(v)|^2 \right) dv e^{-\delta\sqrt{n}t} \end{aligned}$$

where $K := \max\{2A+1; 2C+1\}$. \square

3.3 With $\gamma = 1$ and Dirichlet boundary conditions

The key point of the proof of Theorem 2 for the Kolmogorov equation with $\gamma = 1$ and Dirichlet boundary conditions is the following result.

Proposition 15. *We assume $\gamma = 1$. There exists $K, \delta > 0$ such that, for every $n \in \mathbb{Z} - \{0\}$ and $g_{0,n} \in H^1(-1, 1)$, the solution of (40) satisfies*

$$\|g_n(t)\|_{L^2(-1,1)} \leq Ke^{-\delta|n|^{2/3}t} \|g_{0,n}\|_{H^1(-1,1)}, \quad \forall t > 0. \quad (44)$$

Moreover, the power “2/3” in the exponential rate is optimal as $n \rightarrow +\infty$, and necessarily $\delta \leq \frac{\mu}{2}$, where μ is the first zero (from the right) of Airy function in the half line $(-\infty, 0)$.

The first statement is proved in [4] with a strict Lyapunov function inspired from [43]. The second statement is related to the study of the complex Airy operator performed in [3].

4 Conclusion and open problems

In this article we have studied the null controllability of

- the Grushin type equation (1), in the rectangle $\Omega = (-1, 1) \times (0, 1)$,
- the Kolmogorov equation (2), in the rectangle $\Omega = \mathbb{T} \times (-1, 1)$,

with a distributed control localized on an open subset ω of Ω .

For Grushin-type operators, we have proved that null controllability:

- holds in any positive time, when degeneracy is not too strong, i.e. $\gamma \in (0, 1)$,
- holds only in large time, when $\gamma = 1$ and ω is a strip parallel to the y -axis,

- does not hold when degeneracy is too strong, i.e. $\gamma > 1$.

Null controllability when $\gamma = 1$, T is large enough, and the control region ω is more general is an open problem. When $\gamma = 1$, it would be interesting to characterize the minimal time T^* required. We conjecture that $T^* = \frac{a^2}{2}$. The technique of this paper should possibly extend to higher dimensional cylindrical domains of the form $(-1, 1) \times (0, 1)^m$. However, the generalization of this result to other multidimensional configurations (including $x \in (-1, 1)^n$, $y \in (0, 1)^m$ with $m, n \geq 1$) or boundary controls, is widely open.

For Kolmogorov-type equations, we have proved that null controllability:

- holds in any positive time, with $\gamma = 1$ and Dirichlet boundary conditions in v ,
- holds in any positive time, when $\gamma = 1$, ω is a strip parallel to the x -axis and with Dirichlet boundary conditions in v ,
- holds only in large time, when $\gamma = 2$ and $\omega = \mathbb{T} \times (a, b)$, $0 < a < b < 1$.

The following questions are still open.

1. When $\gamma > 2$, does null controllability hold? In [5], the proof of the non uniform observability relies on a comparison argument (maximum principle), which cannot be used here because the 1D heat equation has complex valued coefficients.
2. When $\gamma = 2$, what is the value of the minimal time T^* ? We conjecture that $T^* = a^2/2$.
3. With $\gamma = 1$ and Dirichlet boundary conditions in v , does null controllability hold with an arbitrary control support ω ?
4. Is it possible to extend these results to multidimensional configurations? The technique of this paper should possibly extend to cylindrical domains of the form $\mathbb{T} \times (-1, 1)^m$. However, the generalization to more general configurations or boundary controls, is widely open.

References

- [1] F. Alabau-Boussouira, P. Cannarsa, and G. Fragnelli. Carleman estimates for degenerate parabolic operators with applications to null controllability. *J. Evol. Equ.*, 6(2):161–204, 2006.
- [2] S. Alinhac and C. Zuily. Uniqueness and nonuniqueness of the cauchy problem for hyperbolic operators with double characteristics. *Comm. Partial Differential Equations*, 6(7):799–828, 1981.
- [3] Y. Almog. The stability of the normal state of superconductors in the presence of electric currents.
- [4] K. Beauchard. Null controllability of kolmogorov-type equations. (*preprint*), 2012.

- [5] K. Beauchard, P. Cannarsa, and R. Guglielmi. Null controllability of Grushin-type operators in dimension two. (*preprint*), 2011.
- [6] K. Beauchard and E. Zuazua. Some controllability results for the 2D Kolmogorov equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26:1793–1815, 2009.
- [7] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau. Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem. *J. Math. Anal. Appl.*, 336(2):865–887, 2007.
- [8] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau. On the controllability of linear parabolic equations with an arbitrary control location for stratified media. *C. R. Math. Acad. Sci. Paris*, 344(6):357–362, 2007.
- [9] P. Cannarsa and L. de Teresa. Controllability of 1-d coupled degenerate parabolic equations. *Electron. J. Differ. Equ.*, Paper No. 73:21 p., 2009.
- [10] P. Cannarsa, G. Fragnelli, and D. Rocchetti. Null controllability of degenerate parabolic operators with drift. *Netw. Heterog. Media*, 2(4):695–715 (electronic), 2007.
- [11] P. Cannarsa, G. Fragnelli, and D. Rocchetti. Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form. *J. Evol. Equ.*, 8:583–616, 2008.
- [12] P. Cannarsa, P. Martinez, and J. Vancostenoble. Persistent regional null controllability for a class of degenerate parabolic equations. *Commun. Pure Appl. Anal.*, 3(4):607–635, 2004.
- [13] P. Cannarsa, P. Martinez, and J. Vancostenoble. Null controllability of degenerate heat equations. *Adv. Differential Equations*, 10(2):153–190, 2005.
- [14] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates for a class of degenerate parabolic operators. *SIAM J. Control Optim.*, 47(1):1–19, 2008.
- [15] P. Cannarsa, P. Martinez, and J. Vancostenoble. Carleman estimates and null controllability for boundary-degenerate parabolic operators. *C. R. Math. Acad. Sci. Paris*, 347(3-4):147–152, 2009.
- [16] A. Doubova, E. Fernández-Cara, and E. Zuazua. On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. *SIAM J. Control Optim.*, 42 (3):798–819, 2002.
- [17] A. Doubova, A. Osses, and J.-P. Puel. Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients. *ESAIM Control Optim. Calc. Var.*, 8:621–661, 2002.
- [18] T. Duyckaerts, X. Zhang, and E. Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré : Analyse Nonlinéaire*, 25:1–41, 2008.

- [19] Sylvain Ervedoza. Control and stabilization properties for a singular heat equation with an inverse-square potential. *Comm. Partial Differential Equations*, 33(10-12):1996–2019, 2008.
- [20] C. Fabre, J.P. Puel, and E. Zuazua. Approximate controllability of the semilinear heat equation. *Proc. Roy. Soc. Edinburgh*, 125A:31–61, 1995.
- [21] H.O. Fattorini and D. Russel. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.
- [22] E. Fernández-Cara and E. Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17:583–616, 2000.
- [23] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: The linear case. *Advances in Differential Equations*, 5(4-6):465–514, 2000.
- [24] E. Fernández-Cara and E. Zuazua. On the null controllability of the one-dimensional heat equation with BV coefficients. *Computational and Applied Mathematics*, 12:167–190, 2002.
- [25] C. Flores and L. de Teresa. Carleman estimates for degenerate parabolic equations with first order terms and applications. *C. R. Math. Acad. Sci. Paris*, 348(7-8):391–396, 2010.
- [26] A.V. Fursikov and O.Y. Imanuvilov. Controllability of evolution equations. *Lecture Notes Series, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul*, 34, 1996.
- [27] N. Garofalo. Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension. *J. Differential Equations*, 104(1):117–146, 1993.
- [28] M. González-Burgos and L. de Teresa. Some results on controllability for linear and nonlinear heat equations in unbounded domains. *Adv. Differential Equations*, 12 (11):1201–1240, 2007.
- [29] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.
- [30] O.Y. Imanuvilov. Boundary controllability of parabolic equations. *Uspekhi. Mat. Nauk*, 48(3(291)):211–212, 1993.
- [31] O.Y. Imanuvilov. Controllability of parabolic equations. *Mat. Sb.*, 186(6):109–132, 1995.
- [32] O.Y. Imanuvilov and M. Yamamoto. Carleman estimate for a parabolic equation in sobolev spaces of negative order and its applications. *Control of Nonlinear Distributed Parameter Systems, G. Chen et al. eds., Marcel-Dekker*, pages 113–137, 2000.
- [33] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. P.D.E.*, 20:335–356, 1995.

- [34] G. Lebeau and J. Le Rousseau. On Carleman estimates for elliptic and parabolic operators. applications to unique continuation and control of parabolic equations.
- [35] J.-L. Lions. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Avant propos de P. Lelong. Dunod, Paris, 1968.
- [36] A. Lopez and E. Zuazua. Uniform null controllability for the one dimensional heat equation with rapidly oscillating periodic density. *Annales IHP. Analyse non linéaire*, 19 (5):543–580, 2002.
- [37] P. Martinez and J. Vancostenoble. Carleman estimates for one-dimensional degenerate heat equations. *J. Evol. Equ.*, 6(2):325–362, 2006.
- [38] P. Martinez, J. Vancostenoble, and J.-P. Raymond. Regional null controllability of a linearized Crocco type equation. *SIAM J. Control Optim.*, 42, no. 2:709–728, 2003.
- [39] L. Miller. On the null-controllability of the heat equation in unbounded domains. *Bulletin des Sciences Mathématiques*, 129, 2:175–185, 2005.
- [40] L. Miller. On exponential observability estimates for the heat semigroup with explicit rates. *Rendiconti Lincei: Matematica e Applicazioni*, 17, 4:351–366, 2006.
- [41] J. Le Rousseau. Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients. *J. Differential Equations*, 233(2):417–447, 2007.
- [42] J. Vancostenoble and E. Zuazua. Null controllability for the heat equation with singular inverse-square potentials. *J. Funct. Anal.*, 254(7):1864–1902, 2008.
- [43] C. Villani. *Hypo-coercivity.*, volume 202. Mem. Amer. Math. Soc., 2009.
- [44] Zabczyk. *Mathematical control theory: an introduction*. Birkhäuser, 2000.
- [45] E. Zuazua. Approximate controllability of the semilinear heat equation: boundary control. *International Conference in honour of Prof. R. Glowinski, Computational Sciences for the 21st Century, M.O. Bristeau et al. eds., John Wiley and Sons*, pages 738–747, 1997.
- [46] E. Zuazua. Finite dimensional null-controllability of the semilinear heat equation. *J. Math. Pures et Appl.*, 76:237–264, 1997.
- [47] C. Zuily. *Uniqueness and non-uniqueness in the Cauchy problem*. Boston Basel Stuttgart Birkhäuser, 1983.